NOTE: The questions on this exam does not exactly reflect which questions will be on this terms exam. That is, some questions asked on this exam may not be asked on our exam and there may be some questions on our exam not asked here.

1. Short Answer Problems

a) Let $A = \begin{bmatrix} 3+i & 2 \\ i & 1 \end{bmatrix}$. Compute A^*.

b) Let $A = \begin{bmatrix} 3 & i \\ -i & 2 \end{bmatrix}$. Is A normal?

c) Show that the product of two orthogonal matrices is an orthogonal matrix.

d) Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$. Find an orthogonal matrix P such that P^TAP is in real canonical form.

2. Let W be the vector space of all 2×2 upper triangular matrices with real entries. So

$$W = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}.$$

Consider the linear mapping $L : \mathbb{R}^3 \to W$ defined by

$$L(x, y, z) = \begin{bmatrix} x+y & x+z \\ 0 & y-z \end{bmatrix}.$$

a) Find the rank and nullity of L.

b) Explain why L is not an isomorphism. Explain why $\mathbb{R}^3$ and W are isomorphic without finding an isomorphism between them.

c) Find the matrix of L with respect to the standard basis, S, for $\mathbb{R}^3$ and the basis

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

for W.

3. Let V be an inner product space, with inner product $\langle\, , \rangle$. Let $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ be an orthogonal basis for the subspace W of V.

a) Define $W^\perp$, the orthogonal complement of W.

b) Show that $W \cap W^\perp = \{\vec{0}\}$.

c) Let $\vec{x} \in V$. Prove that there exists a unique $\vec{p} \in W$ and $\vec{r} \in W^\perp$ such that $\vec{x} = \vec{p} + \vec{r}$.

d) Prove that $\vec{p}$ is the closest vector in W to $\vec{x}$.

4. Let $A = \begin{bmatrix} 2 & 5 \\ 5 & 2 \end{bmatrix}$ and $Q(\vec{x}) = \vec{x}^T A\vec{x}$.

a) By diagonalizing A, express $Q(\vec{x})$ in diagonal form and give an orthogonal matrix that diagonalizes A. Classify $Q(x, y)$.

b) Sketch the graph of $Q(\vec{x}) = 21$ showing both the original and new axes. Find the equation of the asymptotes.

5. Use the method of least squares to obtain the best straight line solutions $y = mx + c$ for the data:

x	-2	-1	1	2
y	2	3	1	2

6. Find a singular value decomposition of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$.

7. Consider $\mathbb{C}^3$ with its standard inner product. Let $\vec{z} = \begin{bmatrix} 1+i \\ 2-i \\ -1+i \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1-i \\ 2+i \\ 3 \end{bmatrix}$.

a) Evaluate $\langle \vec{z}, \vec{w} \rangle$ and $\langle \vec{w}, 2i\vec{z} \rangle$.

b) Find a vector in span$\{\vec{z}, \vec{w}\}$ that is orthogonal to $\vec{z}$.

8. Let V be an inner product space, with complex inner product $\langle\, ,\, \rangle$.

a) Prove that if $< \vec{u}, \vec{v} >= 0$, then $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$.

b) Prove that if $\vec{u}, \vec{v} \in \mathbb{R}^n$, then the converse is true, but if $\vec{u}, \vec{v} \in \mathbb{C}^n$, then the converse is not necessarily true.

9. Let $H = A + Bi$ be an $n \times n$ matrix where A and B are real $n \times n$ matrices. Prove that H is Hermitian if and only if A is symmetric and B is skew-symmetric.

10. Let c be a non-zero real number and let $A = \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix}$. Find a 2×2 unitary matrix U such that U^*AU is diagonal.

11. Let $\langle\, ,\, \rangle$ denote the standard inner product on $\mathbb{C}^n$ and let U be an $n \times n$ unitary matrix.

a) Show that $\langle U\vec{z}, U\vec{w} \rangle = \langle \vec{z}, \vec{w} \rangle$ for any $\vec{z}, \vec{w} \in \mathbb{C}^n$.

b) Suppose λ is an eigenvalue of U. Use part a) to show that $|\lambda| = 1$.