Math 235 **Final F09 Answers**

1. Short Answer Problems

a) $A^* = \begin{bmatrix} 3-i & -i \\ 2 & 1 \end{bmatrix}$.

b) A is Hermitian since $A^* = A$, and hence A is normal.

c) Let Q and P be orthogonal matrices so that $Q^{-1} = Q^T$ and $P^{-1} = P^T$. Then,

$$(PQ)^{-1} = Q^{-1}P^{-1} = Q^T P^T = (PQ)^T.$$

Hence, PQ is orthogonal.

d) Observe that the characteristic polynomial of A is $C(\lambda) = (3-\lambda)(\lambda^2 - 4\lambda + 5)$. Hence, the eigenvalues of A are $\lambda = 3$ and $\lambda = 2 \pm i$. Thus, A is already in real canonical form with $P = I$.

2. a) nullity $L = 1$ and rank $L = 2$.

b)Since null(L) is non-trivial, L is not one-to-one and hence it is not an isomorphism. On the other hand, since $\dim(\mathbb{R}^3) = \dim(W) = 3$, we have that $\mathbb{R}^3$ and W are isomorphic.

c) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

3. a) $W^\perp = \{\vec{x} \in V \mid \langle \vec{x}, \vec{w} \rangle = 0 \text{ for every } \vec{w} \in W\}$.

b) Assume that $\vec{v} \in W$ and $\vec{v} \in W^\perp$. By definition of $W^\perp$ we have that $\langle \vec{v}, \vec{s} \rangle = 0$ for every $\vec{s} \in W$. But, $\vec{v} \in W$, so we have $\langle \vec{v}, \vec{v} \rangle = 0$ and hence $\vec{v} = \vec{0}$ since $\langle \, , \, \rangle$ is an inner product.

c) Let $\{\vec{v}_{n+1}, \dots, \vec{v}_k\}$ be a basis for $W^\perp$. Then we know that $\{\vec{v}_1, \dots, \vec{v}_n, \vec{v}_{n+1}, \dots, \vec{v}_r\}$ is a basis for V. Hence, for any $\vec{x} \in V$ there exists unique coefficients $c_1, \dots, c_r$ such that

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n + c_{n+1}\vec{v}_{n+1} + \cdots + c_r\vec{v}_r = \vec{p} + \vec{r}$$

where $\vec{p} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ and $\vec{r} = c_{n+1}\vec{v}_{n+1} + \cdots + c_r\vec{v}_r$.

d) Let $\vec{w}$ be any vector in W, $\vec{w} \neq \operatorname{proj}_W \vec{x}$. Consider $\vec{x} - \vec{w} = (\vec{x} - \operatorname{proj}_W \vec{x}) + (\operatorname{proj}_W \vec{x} - \vec{w})$. Then

$$\|\vec{x}-\vec{w}\|^2 = \|(\vec{x}-\operatorname{proj}_W \vec{x}) + (\operatorname{proj}_W \vec{x}-\vec{w})\|^2 = \|\vec{x}-\operatorname{proj}_W \vec{x}\|^2 + \|\operatorname{proj}_W \vec{x}-\vec{w}\|^2 > \|\vec{x}-\operatorname{proj}_W \vec{x}\|^2.$$

Thus $\vec{p} = \operatorname{proj}_W \vec{x}$ is the closest vector in W to $\vec{x}$.

4. a) $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, and $Q = 7x_1^2 - 3y_1^2$. Q is indefinite.

b) The asymptotes in the xy-plane are $y = \frac{1-\sqrt{7/3}}{1+\sqrt{7/3}}x$, and $y = \frac{1+\sqrt{7/3}}{1-\sqrt{7/3}}x$.

5. $y = -0.2x + 2$.

6. $A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$

7. a) $\langle \vec{z}, \vec{w} \rangle = i$, $\langle \vec{w}, 2i\vec{z} \rangle = -2$.

b) $\left(\frac{8}{9} - \frac{8}{9}i, \frac{19}{9} + \frac{11}{9}i, \frac{26}{9} - \frac{1}{9}i\right)$

8. a) We have

$$\begin{aligned} \|\vec{u}+\vec{v}\|^2 &= <\vec{u}+\vec{v}, \vec{u}+\vec{v}> = <\vec{u}, \vec{u}+\vec{v}> + <\vec{v}, \vec{u}+\vec{v}> \\ &= <\vec{u},\vec{u}> + <\vec{u},\vec{v}> + <\vec{v},\vec{u}> + <\vec{v},\vec{v}> \\ &= \|\vec{u}\|^2 + 0 + \bar{0} + \|\vec{v}\|^2 \\ &= \|\vec{u}\|^2 + \|\vec{v}\|^2 \end{aligned}$$

b) If $\vec{u}, \vec{v} \in \mathbb{R}^n$, then from our work in a) we have

$\|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{u}^2 + \vec{v}\|^2 = \|\vec{u}\|^2 + <\vec{u},\vec{v}> + <\vec{v},\vec{u}> + \|\vec{v}\|^2 = \|\vec{u}\|^2 + 2<\vec{u},\vec{v}> + \|\vec{v}\|^2.$

Hence $2<\vec{u},\vec{v}> = 0$ and $<\vec{u},\vec{v}> = 0$ as required.

Consider $\vec{u} = 1+i$ and $\vec{v} = 1-i$. Then $\|\vec{u}+\vec{v}\|^2 = \|2\|^2 = 4$ and $\|\vec{u}\|^2 + \|\vec{v}\|^2 = 2+2 = 4$, but $<\vec{u},\vec{v}> = (1+i)(1+i) = 2i \neq 0$.

9. If H is Hermitian, then $H = H^*$ and so

$$A + Bi = (A+Bi)^* = A^* - iB^*.$$

Comparing real and imaginary parts we get $A = A^* = A^T$ and $B = -B^* = -B^T$, since A and B are real. Hence A is symmetric and B is skew-symmetric.

If A is symmetric and B is skew-symmetric, then since A and B are real we get

$$A + Bi = A^T - iB^T = A^* - iB^* = (A+Bi)^*$$

and hence $H = A + Bi$ is Hermitian.

10. $U = \begin{bmatrix} \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, $D = \begin{bmatrix} ic & 0 \\ 0 & -ic \end{bmatrix}$.

11. a) We know that $\langle v, w \rangle = v^T \bar{w}$, or equivalently $\langle v, w \rangle = \overline{\langle w, v \rangle} = \bar{w}^T v = w^* v$. Using this, we have $\langle Uz, Uw \rangle = (Uw)^*(Uv) = w^* U^* U v = w^* v = \langle v, w \rangle$, using the fact that $U^* U = I$.

b) Suppose λ is an eigenvalue of U. Then $Uv = \lambda v$ for some $v \in \mathbb{C}^n$. We get $||Uv||^2 = \langle Uv, Uv \rangle = \langle v, v \rangle = ||v||^2$ by part (a). But we also have $||Uv||^2 = ||\lambda v||^2 = \langle \lambda v, \lambda v \rangle = \lambda\bar{\lambda}\langle v, v \rangle = |\lambda|^2 ||v||^2$, so since $||v|| \neq 0$, we must have $|\lambda| = 1$.