NOTE: The questions on this exam does not exactly reflect which questions will be on this terms exam. That is, some questions asked on this exam may not be asked on our exam and there may be some questions on our exam not asked here.

1. Short Answer Problems

a) By considering the dimensions of the range or null space, determine the rank and the nullity of $L : P_2 \to M(2,2)$ given by $L(ax^2 + bx + c) = \begin{bmatrix} a & a \\ c & c \end{bmatrix}$.

b) Let V, W be finite dimensional vectors spaces over $\mathbb{R}$. Give the formula for finding the matrix of a linear transformation $L : V \to W$ with respect to any basis B for V and any basis C for W.

c) Let A and B be $n \times n$ real matrices such that $A = A^T$ and $B = -B^T$. Prove that $A + iB$ is Hermitian.

d) State the principal axis theorem.

e) State Schur's theorem.

2. Let $\vec{y} = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix}$ and let W be the subspace of $\mathbb{R}^3$ spanned by $\left\{ \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\}$.

a) Find $\text{proj}_W(\vec{y})$.

b) Find the distance from $\vec{y}$ to W.

3. Let $<,>$ be any inner product on a real vector space V and let W be any subspace of V. Prove that for any $\vec{v} \in V$ we have the Pythagorean theorem

$$\|\vec{v}\|^2 = \|\text{proj}_W \vec{v}\|^2 + \|\vec{v} - \text{proj}_W \vec{v}\|^2$$

4. Consider the quadratic form $Q(x, y) = 7x^2 + 12xy + 12y^2$.

a) Find the symmetric matrix A that corresponds to Q and, by diagonalizing A, express $Q(x, y)$ in diagonal form giving an orthogonal matrix that diagonalizes A. Classify Q.

b) Sketch $Q(x, y) = 48$ showing both the original axis and the new axis.

5. Let A and B be symmetric $n \times n$ matrices whose eigenvalues are all positive. Show that the eigenvalues of $A + B$ are all positive.

6. Find a and b to obtain the best fitting equation of the form $y = a + bt$ for the given data.

t	-2	-1	0	1	2
y	2	5	6	9	11

7. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

a) Find the singular value decomposition of A.

b) Find the real canonical form of A and give a change of basis matrix P that brings the matrix into this form.

8. Use the Gram-Schmidt procedure to produce an orthogonal basis from the basis $B = \left\{ \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \begin{bmatrix} i \\ 0 \\ i \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ i \end{bmatrix} \right\}$ of $\mathbb{C}^3$ under the standard complex inner product.

9. Let A and B be Hermitian matrices. Prove that AB is Hermitian if and only if $AB = BA$.

10. Suppose that A is similar to $B = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$.

a) Prove that A is diagonalizable.

b) Prove that $(A - I)(A - 2I)(A - 3I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

11. Prove that every normal matrix N is unitarily diagonalizable.

12. Unitarily diagonalize $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1+i \\ 0 & 1-i & 2 \end{bmatrix}$.

13. Determine if each statement is True or False. Justify each answer.

a) If $H = A + Bi$ is a Hermitian matrix where A and B are real matrices, then A is symmetric and B is skew-symmetric.

b) $\begin{bmatrix} 1+i & 2i \\ i & 3 \end{bmatrix}$ is unitarily diagonalizable.

c) If two $n \times n$ matrices are unitarily similar, then they have the same eigenvalues.

d) If P is a real, orthogonal matrix, then $\det P = \pm 1$.

e) If A is a 2×2, real, orthogonal, symmetric matrix, then $< \vec{x}, \vec{y} >= \vec{x}^T A \vec{y}$ is an inner product for $\mathbb{R}^2$.