Math 235 **Final S09 Answers**

1. a) A basis for the nullspace is $\{x\}$, hence the nullity of L is 1. Thus, by the dimension theorem we have $\operatorname{rank} L + 1 = \dim P_2 = 3$, hence $\operatorname{rank} L = 2$.

b) Let $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ then $[L] = \begin{bmatrix} [L(\vec{v}_1)]_C & \cdots & [L(\vec{v}_n)]_C \end{bmatrix}$

c) If $A\vec{v} = \lambda\vec{v}$, with $\vec{v} \neq \vec{0}$ then $\vec{v}$ is an eigenvector of A with eigenvalue λ.

d) A is symmetric if and only if A is orthogonally diagonalizable.

e) State Schur's theorem.
Every square matrix is unitarily similar to a upper triangular matrix T where the diagonal entries of T are the eigenvalues of A.

2. a) $\operatorname{proj}_W \vec{y} = (-3, -5, 1) + (6, -4, -2) = (3, -9, -1)$

b) $\sqrt{40}$.

3.

$$\begin{aligned}
\| \operatorname{proj}_W \vec{v}\|^2 + \|\vec{v} - \operatorname{proj}_W \vec{v}\|^2 &= < \operatorname{proj}_W \vec{v}, \operatorname{proj}_w \vec{v} > + < \vec{v} - \operatorname{proj}_w \vec{v}, \vec{v} - \operatorname{proj}_W \vec{v} > \\
&= < \operatorname{proj}_W \vec{v}, \operatorname{proj}_W \vec{v} > + < \vec{v}, \vec{v} - \operatorname{proj}_W \vec{v} > - < \operatorname{proj}_W \vec{v}, \vec{v} - \operatorname{proj}_W \vec{v} > \\
&= < \operatorname{proj}_W \vec{v}, \operatorname{proj}_W \vec{v} > + < \vec{v}, \vec{v} - \operatorname{proj}_W \vec{v} > -0 \\
&= < -\operatorname{proj}_W \vec{v}, -\operatorname{proj}_W \vec{v} > + < \vec{v}, \vec{v} > + (\vec{v}, -\operatorname{proj}_W \vec{v} > \\
&= < \vec{v}, \vec{v} > - < \vec{v} - \operatorname{proj}_W \vec{v}, -\operatorname{proj}_W \vec{v} > \\
&= \|\vec{v}\|^2 - 0
\end{aligned}$$

4. a) $Q(x_1, y_1) = 3x_1^2 + 16y_1^2$, Q is positive definite, $P = \begin{bmatrix} -3/\sqrt{13} & 2/\sqrt{13} \\ 2/\sqrt{13} & 3/\sqrt{13} \end{bmatrix}$.

5. Since the eigenvalues of A and B are all positive we have that A and B are positive definite hence $\vec{x}^T A\vec{x} > 0$ and $\vec{x}^T B\vec{x} > 0$ for all $vx \neq \vec{0}$. Hence $\vec{x}^T(A+B)\vec{x} = \vec{x}^T A\vec{x} + \vec{x}^T B\vec{x} > 0$ for all $vx \neq \vec{0}$ so $A + B$ is positive definite and thus has all positive eigenvalues.

6. $y = 6.6 + 2.2t$.

7. a) $A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

b) A is already in real canonical form with $\lambda = 1 - i$ and $P = I$.

8. $\vec{v}_1 = (1, i, 0)$, $\vec{v}_2 = (i, 1, 2i)$, $\vec{v}_3 = (\frac{1}{3}i, \frac{1}{3}, -\frac{1}{3}i)$

9. If AB is Hermitian, then $(AB) = (AB)^* = B^*A^* = BA$, since A and B are Hermitian. If $AB = BA$, then $(AB)^* = B^*A^* = BA = AB$.

10. Let λ be an eigenvalue of A with unit eigenvector $\vec{z}$. Then,

$$\begin{aligned}\lambda = \lambda < \vec{z}, \vec{z} > &= < \lambda\vec{z}, \vec{z} > = < A\vec{z}, \vec{z} > \\ &= < \vec{z}, A\vec{z} > = < \vec{z}, \lambda\vec{z} > = \overline{\lambda} < \vec{z}, \vec{z} > = \overline{\lambda}\end{aligned}$$

Hence $\lambda = \overline{\lambda}$ so λ is real.

11. By Schur's theorem, there exist an upper triangular matrix T and unitary matrix U such that $U^*AU = T$. We just need to prove that T is in fact diagonal. Observe that

$$TT^* = (U^*AU)(U^*A^*U) = U^*AA^*U = U^*A^*AU = (U^*A^*U)(U^*AU) = T^*T.$$

Hence T is also normal and if we compare the diagonal entries of TT^* and T^*T we get

$$\begin{aligned}|t_{11}|^2 + |t_{12}|^2 + \cdots + |t_{1n}|^2 &= |t_{11}|^2 \\ |t_{22}|^2 + \cdots + |t_{2n}|^2 &= |t_{12}|^2 + |t_{22}|^2 \\ &\vdots \\ |t_{nn}|^2 &= |t_{1n}|^2 + |t_{2n}|^2 + \cdots + |t_{nn}|^2\end{aligned}$$

Hence, we must have $t_{ij} = 0$ for all $i \neq j$ and hence T is diagonal as required.

12. $U = \begin{bmatrix} 0 & 1 & 0 \\ (1+i)/\sqrt{3} & 0 & (1+i)/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

13. a) TRUE. $H^* = (A+Bi)^* = A^* - iB^* = A + Bi$. So $A^T = A^* = A$ and $B^T = B^* = -B$.

b) FALSE. Not normal.

c) TRUE. Let $A = U^*BU$ then $\det(A - \lambda I) = \det(U^*)\det(B - \lambda I)\det(U) = \det(B - \lambda I)$.

d) TRUE. $PP^T = I \Rightarrow \det(PP^T) = 1 \Rightarrow \det P = \pm 1$.

e) FALSE. A needs to be positive definite.