Math 235 **Sample Term Test 1 - 1**

1. Short Answer Problems

a) Give the definition of an inner product $\langle\ ,\rangle$ on a vector space V.

b) Let $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ be orthonormal in an inner product space V and let $\vec{v} \in V$ such that $\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$. Prove that $a_i =< \vec{v}, \vec{v}_i >$.

c) Define what it means for a set B to be orthonormal in an inner product space V.

d) State the Rank-Nullity Theorem.

e) Find the rank and nullity of the linear mapping $L : \mathbb{R}^3 \to M(2, 2)$ defined by

$$L(x_1, x_2, x_3) = \begin{bmatrix} x_1 & x_1 + x_2 \\ x_2 & x_1 - x_2 \end{bmatrix}.$$

2. Let V be an n-dimensional vector space, and let $T : V \to V$ be defined by $T(v) = \lambda v$ for all v in V, where $\lambda \in \mathbb{R}$ is a constant.

a) Prove that T is linear.

b) Compute the nullspace and the range of T. There are two cases, depending on λ.

c) Let $\beta = \{v_1, \dots, v_n\}$ be a basis for V. Give the matrix $[T]_\beta$ for the map T with respect to the basis β.

3. Find the matrix of $L : \mathbb{R}^2 \to P_2$ defined by $L(a_1, a_2) = a_1x^2 + (a_1 + a_2)$ with respect to the basis $B = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ of $\mathbb{R}^2$ and $C = \{x^2 + 1, x + 1, x^2 - x - 1\}$ of P_2.

4. Let A be an $m \times n$ matrix. Prove that $A\vec{x} = \vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^m$ if and only if the equation $A^T\vec{y} = \vec{0}$ has only the trivial solution.

5. Let V be a vector space of dimension n. Prove that the vector space of all linear operators from V to V is isomorphic to $M(n, n)$.

6. Let $<\,,>$ be the standard inner product in $\mathbb{R}^n$ and let U be an $m \times n$ matrix with orthonormal columns. Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Prove that $<\vec{x}, \vec{y}> = <U\vec{x}, U\vec{y}>$ and thus that $\|U\vec{x}\| = \|\vec{x}\|$.

7. Let $B = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and $C = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and define $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$[\vec{x}]_B = [L(\vec{x})]_C.$$

a) Find $L \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

b) Find $L \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

c) Prove that L is an isomorphism.

8. Let $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\} \in \mathbb{R}^2$ and define

$$< \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} > = x_1 x_2 + 8 y_1 y_2 + 2 x_1 y_2 + 2 y_1 x_2.$$

a) Prove that $<\,,>$ defines an inner product on $\mathbb{R}^2$.

b) Show that B is an orthogonal basis for $\mathbb{R}^2$ using this inner product and produce an orthonormal basis.

c) Find the B-coordinates of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$