**Math 235** **Sample Term Test 1 - 1 Answers**

**NOTE**: - Only answers are provided here (and some proofs). On the test you **must** provide full and complete solutions to receive full marks.

**1.** Short Answer Problems

a) Give the definition of an inner product $\langle\ ,\rangle$ on a vector space $V$.

Solution: $\langle\ ,\rangle : V \times V \to \mathbb{R}$ such that
$\langle \vec{v}, \vec{v}\rangle > 0$ for all $\vec{v} \neq \vec{0}$ and $\langle \vec{0}, \vec{0}\rangle = 0$.
$\langle \vec{v}, \vec{w}\rangle = \langle \vec{w}, \vec{v}\rangle$
$\langle \vec{v}, a\vec{w} + b\vec{x}\rangle = a\langle \vec{v}, \vec{w}\rangle + b\langle \vec{v}, \vec{x}\rangle$

b) Let $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ be orthonormal in an inner product space $V$ and let $\vec{v} \in V$ such that $\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$. Prove that $a_i =< \vec{v}, \vec{v}_i >$.

Solution: Taking the inner product of both sides with $\vec{v}_i$ to get
$< \vec{v}, \vec{v}_i >=< a_1\vec{v}_1 + \cdots + a_n\vec{v}_n, \vec{v}_i >= a_1 < \vec{v}_1, \vec{v}_i > + \cdots + a_n < \vec{v}_n, \vec{v}_i >= a_i$
since $B$ is orthonormal.

c) Define what it means for a set $B$ to be orthonormal in an inner product space $V$.

Solution: A set $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ is orthonormal in $V$ if $< \vec{v}_i, \vec{v}_j >= 0$ for all $i \neq j$ and $< \vec{v}_i, \vec{v}_i >= 1$ for $1 \leq i \leq n$.

d) State the Rank-Nullity Theorem.

Solution: Suppose that $V$ is an $n$-dimensional vector space and that $L : V \to W$ is a linear mapping into a vector space $W$. Then $\text{rank}(L) + \text{Null}(L) = n$.

e) Find the rank and nullity of the linear mapping $L : \mathbb{R}^3 \to M(2, 2)$ defined by

$$L(x_1, x_2, x_3) = \begin{bmatrix} x_1 & x_1 + x_2 \\ x_2 & x_1 - x_2 \end{bmatrix}.$$

Solution: Consider $L(x_1, x_2, x_3) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then we must have $x_1 = 0 = x_2$. Hence, a basis for the nullspace of $L$ is $\{(0, 0, 1)\}$ and so $\text{Null}(L) = 1$. Thus, by the Rank-Nullity theorem we have

$$\text{rank}(L) = \dim \mathbb{R}^3 - (L) = 2.$$

**2.** Let $V$ be an $n$-dimensional vector space, and let $T : V \to V$ be defined by $T(v) = \lambda v$ for all $v$ in $V$, where $\lambda \in \mathbb{R}$ is a constant.

a) Prove that $T$ is linear.

Solution: Let $\vec{x}, \vec{y} \in V$, $k \in \mathbb{R}$. Then

$$T(k\vec{x} + \vec{y}) = \lambda(k\vec{x} + \vec{y}) = k\lambda\vec{x} + \lambda\vec{y} = kT(\vec{x}) + T(\vec{y}).$$

Thus $T$ is linear.

b) Compute the nullspace and the range of $T$. There are two cases, depending on $\lambda$.

Solution: Let $\vec{v}$ be in the nullspace of $T$. Then $T(\vec{v}) = \lambda\vec{v} = \vec{0}$.

If $\lambda \neq 0$, the nullspace of $T$ is $\{\vec{0}\}$ and the range of $T$ is all of $V$.

If $\lambda = 0$, the nullspace of $T$ is all of $V$ and the range of $T$ is $\{\vec{0}\}$.

c) Let $\beta = \{v_1, \dots, v_n\}$ be a basis for $V$. Give the matrix $[T]_\beta$ for the map $T$ with respect to the basis $\beta$.

Solution: $[T]_\beta = \lambda I$

**3.** Find the matrix of $L : \mathbb{R}^2 \to P_2$ defined by $L(a_1, a_2) = a_1x^2 + (a_1 + a_2)$ with respect to the basis $B = \{(1, -1), (1, 2)\}$ of $\mathbb{R}^2$ and $C = \{x^2 + 1, x + 1, x^2 - x - 1\}$ of $P_2$.

Solution: ${}_C[L]_B = \begin{bmatrix} 0 & 3 \\ 1 & -2 \\ 1 & -2 \end{bmatrix}$.

**4.** Let $A$ be an $m \times n$ matrix. Prove that $A\vec{x} = \vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^m$ if and only if the equation $A^T\vec{y} = \vec{0}$ has only the trivial solution.

Solution: If $A\vec{x} = \vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^m$, then $\operatorname{rank} A = m$. Hence, $\operatorname{rank} A^T = m$ and so by the Rank-Nullity theorem,

$$\text{Null}(A^T) = m - \operatorname{rank} A = 0.$$

Therefore, $A^T\vec{y} = \vec{0}$ has only the trivial solution.

If $A^T\vec{y} = \vec{0}$ has only the trivial solution, then $\text{Null}(A^T) = 0$ and so by the Rank-Nullity theorem

$$\operatorname{rank} A^T = m - \text{Null}(A^T) = m.$$

Hence, $\operatorname{rank} A = m$ and so $A\vec{x} = \vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^m$.

**5.** Let $V$ be a vector space of dimension $n$. Prove that the vector space of all linear operators from $V$ to $V$ is isomorphic to $M(n, n)$.

Solution: Let $S$ be the set of all linear operators $L : V \to V$ and let $B$ be a basis for $V$. Define $T : S \to M(n, n)$, by $T(L) = [L]_B$. Prove that $T$ is linear, one-to-one and onto.

**6.** Let $< , >$ be the standard inner product in $\mathbb{R}^n$ and let $U$ be an $m \times n$ matrix with orthonormal columns. Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Prove that $< \vec{x}, \vec{y} > = < U\vec{x}, U\vec{y} >$ and thus that $\|U\vec{x}\| = \|\vec{x}\|$.

Solution: Let $U = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{bmatrix}$ so that $U^T = \begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_n^T \end{bmatrix}$. Then

$$U^TU = \begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_n^T \end{bmatrix} \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{bmatrix} = I$$

since the columns of $U$ are orthonormal. Thus,

$$< U\vec{x}, U\vec{y} > = (U\vec{x})^TU\vec{y} = \vec{x}^TU^TU\vec{y} = \vec{x}^T\vec{y} = < \vec{x}, \vec{y} > .$$

**7.** Let $B = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and $C = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and define $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$[\vec{x}]_B = [L(\vec{x})]_C.$$

a) Find $L \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

Solution: $L(\vec{x}) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

b) Find $L \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution: $L(\vec{x}) = \begin{bmatrix} x_1 \\ -x_1 + x_2 \end{bmatrix}$.

c) Prove that $L$ is an isomorphism.

Solution: Prove that $L$ is linear, one-to-one and onto.

**8.** Let $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\} \in \mathbb{R}^2$ and define

$$< \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} >= x_1x_2 + 8y_1y_2 + 2x_1y_2 + 2y_1x_2.$$

a) Prove that $< , >$ defines an inner product on $\mathbb{R}^2$.

Solution: Show that $< , >$ satisfies the definition of an inner product from 1 a).

b) Show that $B$ is an orthogonal basis for $\mathbb{R}^2$ using this inner product and produce an orthonormal basis.

Solution: We have

$$< \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} > = 0$$

$$< \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} > = 1$$

$$< \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} > = 4$$

Thus, it is an orthogonal basis and $\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1/2 \end{bmatrix} \}$ is an orthonormal basis.

c) Find the $B$-coordinates of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Solution: $\begin{bmatrix} 2 \\ 3 \end{bmatrix}_B = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$