Math 235 **Sample Term Test 1 - 2**

1. Short Answer Problems

a) Write a basis for the rowspace, columnspace and nullspace of $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

b) Let $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be orthonormal in an inner product space V and let $\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$. Prove that $a_i =< \vec{v}, \vec{v}_i >$.

c) State the Rank-Nullity Theorem.

d) Find the rank and nullity of the linear mapping $T : P_2 \to M(2,2)$ defined by

$$T(a + bx + cx^2) = \begin{bmatrix} c & b \\ 0 & c \end{bmatrix}.$$

2. Let $L : M(2,2) \to M(2,2)$ be given by $L(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A^T$. Find the matrix for L relative to the standard basis B of $M(2,2)$, where

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

3. Let $B = \{v_1, v_2\}$ be a basis for V. Let a be a scalar constant. Let $T : V \to V$ be linear and $T(v_1) = av_1 + av_2$, $T(v_2) = 3v_1 - av_2$. For what values of a is T an isomorphism?

4. Let N be the plane with basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$. Define an explicit isomorphism to establish that P_1 and N are isomorphic. Prove that your map is an isomorphism.

5. Let T be a linear operator on an inner product space V, and suppose that $\langle \vec{x}, \vec{y} \rangle = \langle T(\vec{x}), T(\vec{y}) \rangle$ for all $\vec{x}$ and $\vec{y}$ in V. Prove that T is an isomorphism.

6. Let Q be an $n \times n$ orthogonal matrix, and let $\vec{x}$ and $\vec{y}$ be orthogonal vectors in $\mathbb{R}^n$. Show that $Q\vec{x}$ and $Q\vec{y}$ are orthogonal.

7. $B = \left\{ \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^3$.

Using B (or other methods), determine another orthonormal basis for $\mathbb{R}^3$ which includes the vector $\begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{3} \\ 1/\sqrt{2} \end{bmatrix}$, and briefly explain why your basis is orthonormal.

8. Consider P_2 with inner product $\langle p(x), q(x) \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$.

a) Find the value of $\langle 1 - x - x^2, 1 + x^2 \rangle$.

b) Find the distance between $1 - x - x^2$ and $1 + x^2$.

c) Determine the coordinates of $1 - 2x + x^2$ with respect to the orthonormal basis

$$B = \left\{ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}x, \frac{1}{\sqrt{6}}(2 - 3x^2) \right\}.$$

d) Given that $S = \{1 - x^2, \frac{1}{2}(x - x^2)\}$ is orthonormal, extend S to find an orthonormal basis for P_2.

9. Let V be a real inner product space with inner product $\langle\, , \rangle$ and let $\vec{u}, \vec{v} \in V$. Prove that $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$ if and only if $< \vec{u}, \vec{v} >= 0$.

10. Let U, V, W be real vectors spaces and let $L : U \to V$ and $M : V \to W$ be linear mappings. Prove that if L and M are onto, then $M \circ L$ is onto.

11. Let V be an n-dimensional vector space over $\mathbb{R}$ and let S be the vector space of all linear operators $L : V \to V$.

a) Prove that S is isomorphic to $M(n, n)$.

b) Give, with proof, a basis for S.