Math 235 **Sample Term Test 1 - 2 Answers**

NOTE: - Only answers are provided here (and some proofs). On the test you **must** provide full and complete solutions to receive full marks.

1. Short Answer Problems

a) Let $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Write a basis for the Row(A), Col(A) and Null(A).

Solution: A basis for Row(A) is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. A basis for Null($A$) is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

A basis for Col(A) is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

b) Let $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ be orthonormal in an inner product space V and let $\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$. Prove that $a_i = < \vec{v}, \vec{v}_i >$.

Solution: Taking the inner product of both sides with $\vec{v}_i$ to get
$< \vec{v}, \vec{v}_i > = < a_1\vec{v}_1 + \cdots + a_n\vec{v}_n, \vec{v}_i > = a_1 < \vec{v}_1, \vec{v}_i > + \cdots + a_n < \vec{v}_n, \vec{v}_i > = a_i$
since B is orthonormal.

c) State the Rank-Nullity Theorem.

Solution: Suppose that V is an n-dimensional vector space and that $L : V \to W$ is a linear mapping into a vector space W. Then $\text{rank}(L) + \text{Null}(L) = n$.

d) Find the rank and nullity of the linear mapping $T : P_2 \to M(2,2)$ defined by $T(a + bx + cx^2) = \begin{bmatrix} c & b \\ 0 & c \end{bmatrix}$.

Solution: $\text{rank}(T) = 2$, $\text{nullity}(T) = 1$.

2. Let $L : M(2,2) \to M(2,2)$ be given by $L(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A^T$. Find the matrix for L relative to the standard basis B of $M(2,2)$, where $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

Solution: $[L]_B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix}$.

3. Let $B = \{\vec{v}_1, \vec{v}_2\}$ be a basis for V. Let a be a scalar constant. Let $T : V \to V$ be linear and $T(\vec{v}_1) = a\vec{v}_1 + a\vec{v}_2$, $T(\vec{v}_2) = 3\vec{v}_1 - a\vec{v}_2$. For what values of a is T an isomorphism?

Solution: $a \neq 0$ and $a \neq -3$.

4. Let N be the plane in $\mathbb{R}^3$ with basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$. Define an explicit isomorphism to establish that P_1 and N are isomorphic. Prove that your map is an isomorphism.

Solution: Define a mapping $T : P_1 \to N$ where

$$T(a + bx) = a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

To show that T is linear, let $p(x) = a + bx$, $q(x) = c + dx$, and k is a scalar constant. Then

$$\begin{aligned} T(kp(x) + q(x)) &= T((ka + c) + (kb + d)x) \\ &= (ka + c) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + (kb + d) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\ &= k \left(a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right) + \left(c \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + d \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right) \\ &= kT(p(x)) + T(q(x)). \end{aligned}$$

To show that T is one-to-one, we will show that the nullspace of T is $\{0\}$. Suppose that $T(a + bx) = \vec{0}$. Then

$$a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This implies that $a + b = 0$ and $-b = 0$, hence $a = b = 0$. Therefore, the only element of P_1 that is in the nullspace of T is 0.

Now P_1 and N have the same dimension, which is 2. Since T is one-to-one, T is also onto, therefore T is an isomorphism.

5. Let T be a linear operator on an inner product space V, and suppose that $\langle \vec{x}, \vec{y} \rangle = \langle T(\vec{x}), T(\vec{y}) \rangle$ for all $\vec{x}$ and $\vec{y}$ in V. Prove that T is an isomorphism.

Solution: We are given that T is linear. Show T is one-to-one and onto. (Will be solved in the tutorial)

6. Let Q be an $n \times n$ orthogonal matrix, and let $\vec{x}$ and $\vec{y}$ be orthogonal vectors in $\mathbb{R}^n$. Show that $Q\vec{x}$ and $Q\vec{y}$ are orthogonal.

Solution: Since $\vec{x}$ and $\vec{y}$ are orthogonal, $\vec{x} \cdot \vec{y} = 0$. Also, $Q^T Q = I$. Recall that $\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}$. Then

$$(Q\vec{x}) \cdot (Q\vec{y}) = (Q\vec{x})^T (Q\vec{y}) = \vec{x}^T Q^T Q \vec{y} = \vec{x}^T \vec{y} = 0.$$

Therefore, $Q\vec{x}$ and $Q\vec{y}$ are orthogonal.

7. The following is an orthonormal basis for $\mathbb{R}^3$: $B = \left\{ \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \right\}$. Using B (or other methods), determine another orthonormal basis for $\mathbb{R}^3$ which includes the vector $\begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{3} \\ 1/\sqrt{2} \end{bmatrix}$, and briefly explain why your basis is orthonormal.

Solution: The 3×3 matrix whose column vectors are the three vectors in B is orthogonal. The given vector is the first row of this matrix. Since rows of an orthogonal matrix are orthonormal, we can use the rows to form an orthonormal basis for $\mathbb{R}^3$:

$$\left\{ \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{3} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{3} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{3} \\ -1/\sqrt{2} \end{bmatrix} \right\}.$$

8. Consider P_2 with inner product $\langle p(x), q(x) \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$.

a) Find the value of $\langle 1 - x - x^2, 1 + x^2 \rangle$.

Solution: $\langle 1 - x - x^2, 1 + x^2 \rangle = (1)(2) + (1)(1) + (-1)(2) = 1$.

b) Find the distance between $1 - x - x^2$ and $1 + x^2$.

Solution: The distance is $||(1 - x - x^2) - (1 + x^2)|| = \sqrt{(-1)^2 + (0)^2 + (-3)^2} = \sqrt{10}$.

c) Determine the coordinates of $1 - 2x + x^2$ with respect to the orthonormal basis

$$B = \left\{ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}x, \frac{1}{\sqrt{6}}(2 - 3x^2) \right\}.$$

Solution: $[1 - 2x + x^2]_B = \begin{bmatrix} 5/\sqrt{3} \\ -4/\sqrt{2} \\ -2/\sqrt{6} \end{bmatrix}$.

d) Given that $S = \{1 - x^2, \frac{1}{2}(x - x^2)\}$ is orthonormal, extend S to find an orthonormal basis for P_2.

Solution: $\{1 - x^2, \frac{1}{2}(x - x^2), \frac{1}{2}(x + x^2)\}$ is an orthonormal basis of P_2.

9. Let V be a real inner product space with inner product $\langle \, , \rangle$ and let $\vec{u}, \vec{v} \in V$. Prove that $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$ if and only if $< \vec{u}, \vec{v} >= 0$.

Solution: We have

$$\begin{aligned}\|\vec{u}^2+\vec{v}\|^2 &=<\vec{u}+\vec{v},\vec{u}+\vec{v}>=<\vec{u},\vec{u}+\vec{v}>+<\vec{v},\vec{u}+\vec{v}>\\ &=<\vec{u},\vec{u}>+<\vec{u},\vec{v}>+<\vec{v},\vec{u}>+<\vec{v},\vec{v}>\\ &=\|\vec{u}\|^2+0+\overline{0}+\|\vec{v}\|^2\\ &=\|\vec{u}\|^2+\|\vec{v}\|^2\end{aligned}$$

If $\vec{u},\vec{v}\in\mathbb{R}^n$, then we have

$$\|\vec{u}\|^2+\|\vec{v}\|^2=\|\vec{u}^2+\vec{v}\|^2=\|\vec{u}\|^2+<\vec{u},\vec{v}>+<\vec{v},\vec{u}>+\|\vec{v}\|^2=\|\vec{u}\|^2+2<\vec{u},\vec{v}>+\|\vec{v}\|^2.$$

Hence $2<\vec{u},\vec{v}>=0$ and $<\vec{u},\vec{v}>=0$ as required.

10. Let U,V,W be real vectors spaces and let $L:U\to V$ and $M:V\to W$ be linear mappings. Prove that if L and M are onto, then $M\circ L$ is onto.

Solution: To show $M\circ L:U\to W$ is onto, we need to show that for every $\vec{w}\in W$, there exists a $\vec{u}\in U$ such that $(M\circ L)(\vec{u})=\vec{w}$.

Let $\vec{w}\in W$. Then, since M is onto, there exists $\vec{v}\in V$ such that $M(\vec{v})=\vec{w}$. Similarly, since L is onto, there exists $\vec{u}\in U$ such that $L(\vec{u})=\vec{v}$. Thus, we have

$$(M\circ L)(\vec{u})=M(L(\vec{u}))=M(\vec{v})=\vec{w},$$

as required.

11. Let V be an n-dimensional vector space over $\mathbb{R}$ and let S be the vector space of all linear operators $L:V\to V$.

a) Prove that S is isomorphic to $M(n,n)$.

Solution: Let B be a basis for V. Define $T(L)=[L]_B$. Prove this is the desired isomorphism.

b) Give, with proof, a basis for S.

Solution: Let $C=\{\vec{e}_1,\ldots,\vec{e}_{n^2}\}$ be the standard basis for $M(n,n)$. By definition of an isomorphism, T is invertible and T^{-1} is an isomorphism from $M(n,n)\to S$. Moreover, we have that the set

$$D=\{T^{-1}(\vec{e}_1),\ldots,T^{-1}(\vec{e}_{n^2})\}$$

is linearly independent since T^{-1} is one-to-one and C is linearly independent. Thus, since $\dim S=n^2$ as it is isomorphic to $M(n,n)$ we have that D is a set of n^2 linearly independent vectors in S and hence is a basis for S.