Math 235 **Sample Term Test 2 - 1 Answers**

NOTE: - Only answers are provided here (and some proofs). On the test you **must** provide full and complete solutions to receive full marks.

1. Short Answer Problems

a) Let S be a subspace of an inner product space V. What is the definition of $S^\perp$.

Solution: $S^\perp = \{\vec{x} \in V \mid <\vec{x}, \vec{s}> = 0 \text{ for every } \vec{s} \in S\}$.

b) State the Principal Axis Theorem.

Solution: Let A be an $n \times n$ matrix. A is symmetric if and only if A is orthogonally diagonalizable.

c) Determine the matrix for the quadratic form $Q(x, y, z) = 3x^2 - y^2 + z^2 - 2xy + 2yz$.

Solution: $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

d) Write the definition of a quadratic form Q on $\mathbb{R}^n$ being positive definite.

Solution: Q is positive definite if $Q(\vec{x}) > 0$ for all $\vec{x} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$.

e) Let $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ be three orthonormal vectors in $\mathbb{R}^{27}$. Let $\vec{v} = \vec{w}_1 - 4\vec{w}_2 + \sqrt{8}\vec{w}_3$. Compute the length $\|\vec{v}\|$ of $\vec{v}$.

Solution: $\|\vec{v}\| = 5$.

2. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. Find an orthogonal matrix P that diagonalizes A and the corresponding diagonal matrix.

Solution: $P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$, and $P^TAP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

3. Let V be an n-dimensional vector space, and let $\langle\ ,\ \rangle$ be an inner product on V. Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be an orthonormal basis for a subspace S of V, let $P = \text{proj}_S : V \to V$ be the projection onto S.

a) Given an expression for $P(\vec{v})$.

Solution: $P(\vec{v}) = \langle \vec{v}, \vec{v}_1 \rangle \vec{v}_1 + \cdots + \langle \vec{v}, \vec{v}_k \rangle \vec{v}_k$.

b) Show that $P \circ P = P$.

Solution: Since $P(\vec{v}_j) = \vec{v}_j$ for $j = 1 \dots, k$ and $P(\vec{v}_j) = \vec{0}$ for $j = k+1, \dots, n$, we see that

$$P \circ P(\vec{v}_j) = P(P(\vec{v}_j)) = P(\vec{v}_j) = \vec{v}_j \qquad \text{for } j = 1 \dots, k,$$
$$P \circ P(\vec{v}_j) = P(P(\vec{v}_j)) = P(\vec{0}) = \vec{0} \qquad \text{for } j = k+1, \dots, n,$$

and therefore $P \circ P = P$.

c) What are all the eigenvalues and eigenvectors of P?

Solution: 1 has multiplicity k and 0 has multiplicity $n - k$

4. Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{1, x, x^2\}$ be a basis for P_2. Define an inner product $\langle\ ,\ \rangle$ on P_2 by $\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$. Use the Gram-Schmidt process to produce a basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ of P_2 which is orthonormal with respect to this inner product.

Solution: $\left\{\frac{1}{\sqrt{3}}, \frac{x}{\sqrt{2}}, \frac{3}{\sqrt{6}}x^2 - \frac{2}{\sqrt{6}}\right\}$

5. Let A be an $m \times n$ matrix and $\vec{b}$ be an $m \times 1$ column vector. Using the dot product,

a) prove that for any $y \in \mathbb{R}^m$ we have $y \in \big(\mathrm{Col}(A)\big)^\perp$ if and only if $A^T y = \vec{0}$.

Solution: We have $y \in (\mathrm{Col}(A))^\perp$ if and only if the dot product of y with any vector in $\mathrm{Col}(A)$ is zero. But $\mathrm{Col}(A)$ consists of all linear combinations of columns of A. So $y \in (\mathrm{Col}(A))^\perp$ if and only if the dot product of y with any column of A is zero. Now note that the columns of A are the rows of A^T. So $y \in (\mathrm{Col}(A))^\perp$ if and only if the dot product of y with any row of A^T is zero, which is equivalent to $A^T y = 0$.

b) prove that if $\vec{x} \in \mathbb{R}^n$ is the vector which minimizes $||A\vec{x} - \vec{b}||$, then $A^T(A\vec{x} - \vec{b}) = \vec{0}$.

Solution: We have $\{Ax \mid x \in \mathbb{R}^n\}$ is precisely the column space of A. We know the vector in $\mathrm{Col}(A)$ that is closet to b is $\mathrm{proj}_{\mathrm{Col}(A)}\, b$. So if $x \in \mathbb{R}^n$ is the minimizer of $||Ax - b||$ then we must have $Ax = \mathrm{proj}_{\mathrm{Col}(A)}\, b$. Thus $b - Ax = b - \mathrm{proj}_{\mathrm{Col}(A)}\, b = \mathrm{perp}_{\mathrm{Col}(A)}\, b$. Now since $\mathrm{perp}_{\mathrm{Col}(A)}\, b \in (\mathrm{Col}(A))^\perp$ we must have $b - Ax \in (col(A))^\perp$. So by part (a) we have $A^T(Ax - b) = 0$. Rearrange to get $A^T Ax = A^T b$.

c) Find a vector in $\mathrm{Span}\left(\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}\right)$ that is closest to $\begin{bmatrix}1\\1\\2\end{bmatrix}$.

Solution: $Ax = \begin{bmatrix}\frac{5}{3}\\ \frac{1}{3}\\ \frac{4}{3}\end{bmatrix}$.

6. Find a and b to obtain the best fitting equation of the form $y = a + bt$ for the given data.

t	-2	-1	0	1	2
y	2	5	6	9	11

Solution: $y = 6.6 + 2.2t$.

7. Let $A = \begin{bmatrix}6 & 2\\ 2 & 3\end{bmatrix}$ and $Q(\vec{x}) = \vec{x}^T A\vec{x}$.

a) By diagonalizing A, express $Q(\vec{x})$ in diagonal form and give an orthogonal matrix that diagonalizes A. Classify $Q(x, y)$.

Solution: We get $P = \begin{bmatrix}-1/\sqrt{5} & 2/\sqrt{5}\\ 2/\sqrt{5} & 1/\sqrt{5}\end{bmatrix}$, the diagonal form of Q is $Q = 2\tilde{x}^2 + 7\tilde{y}^2$, and Q is positive definite since it has two positive eigenvalues.

b) Sketch the graph of $Q(\vec{x}) = 14$.

8. Prove that every orthogonally diagonalizable matrix is symmetric.

Solution: Assume that A is orthogonally diagonalizable. Then, there exists an orthogonal matrix P such that

$$P^T AP = D,$$

with D diagonal. But, then we have $A = PDP^T$ and

$$A^T = (PDP^T)^T = P^T D^T P^{TT} = P^T DP = A,$$

since D is diagonal. Thus A is symmetric.

9. Let A and B be symmetric $n \times n$ matrices whose eigenvalues are all positive. Show that the eigenvalues of $A + B$ are all positive.

Solution: Since the eigenvalues of A and B are all positive we have that A and B are positive definite hence $\vec{x}^T A\vec{x} > 0$ and $\vec{x}^T B\vec{x} > 0$ for all $\vec{x} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$.

Thus,

$$\vec{x}^T (A + B)\vec{x} = \vec{x}^T A\vec{x} + \vec{x}^T B\vec{x} > 0,$$

for all $\vec{x} \neq \vec{0}$. So $A + B$ is positive definite and thus has all positive eigenvalues.