Math 235 **Sample Term Test 2 - 2 Answers**

NOTE: - Only answers are provided here (and some proofs). On the test you **must** provide full and complete solutions to receive full marks.

1. Short Answer Problems

a) State the Principal Axis Theorem.

Solution: A matrix is symmetric matrix if and only if it is orthogonally diagonalizable.

b) Let A be an $m \times n$ matrix. Prove that A^TA is symmetric.

Solution: $(A^TA)^T = A^TA^{TT} = A^TA$, hence A^TA is symmetric.

c) State the definition of a quadratic form $Q(\vec{x})$ on $\mathbb{R}^n$ being negative definite.

Solution: $Q(\vec{x})$ is negative definite if $Q(\vec{x}) < 0$ for all $\vec{x} \neq \vec{0}$.

d) Consider the quadratic form $Q(\vec{x}) = 3x^2 + 5y^2 + 3z^2 - 2xy - 2xz + 2yz$.
Write down the symmetric matrix A such that $Q(\vec{x}) = \vec{x}^TA\vec{x}$.

Solution: $A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 5 & 1 \\ -1 & 1 & 3 \end{bmatrix}$.

e) For the matrix $A = \begin{bmatrix} -7 & -2 & -3 & -5 & 5 & -1 \\ 0 & 1 & 0 & 2 & -5 & -5 \\ 6 & 4 & 2 & 7 & -10 & -4 \\ 0 & -4 & 0 & -5 & 10 & 10 \\ -6 & -2 & -3 & -5 & 7 & 2 \\ 6 & 2 & 3 & 5 & -5 & 0 \end{bmatrix}$,
-1 is an eigenvalue with multiplicity three, and 2 is an eigenvalue with multiplicity two. Determine the determinant of A.

Solution: $\det A = (-1)(-1)(-1)(2)(2)(-3) = 12$.

2. Let $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. Find an orthogonal matrix P that diagonalizes A and the corresponding diagonal matrix.

Solution: $P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$, and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$.

3. On $M(2,2)$ define the inner product $< A, B >= \mathrm{tr}(A^T B)$.

a) Use the Gram-Schmidt procedure to produce an orthonormal basis for the subspace

$$S = \mathrm{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right\}.$$

Solution: An orthonormal basis is $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$.

b) Consider $A = \begin{bmatrix} 0 & 2 \\ -1 & 2 \end{bmatrix}$. Find the matrix B in S such that $\|A - B\|$ is minimized.

Solution: $B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$.

4. Find b and c to obtain the best fitting equation of the form $y = bx + cx^2$ for the following data:

x	-1	0	1
y	4	1	1

Solution: $y = -\frac{3}{2}x + \frac{5}{2}x^2$.

5. Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$. Find an orthogonal matrix P and upper triangular matrix T such that $P^T AP = T$.

Solution: $P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $T = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$.

6. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $Q(\vec{x}) = \vec{x}^T A\vec{x}$.

a) Find an orthogonal matrix and diagonal matrix D such that $P^T AP = D$.

Solution: $P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

b) Classify $Q(\vec{x})$ as positive definite, negative definite or indefinite.

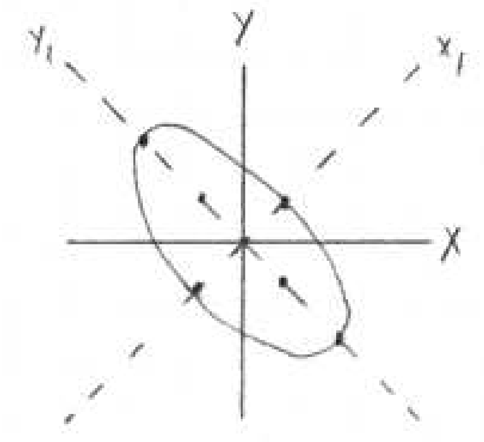
Solution: Since the eigenvalues of A are both positive, $Q(\vec{x})$ is positive definite.

c) Write Q so that it has no cross terms and give the change of variables which brings it into this form.

Solution: Taking $\vec{x} = P\vec{y} = \begin{bmatrix} y_1 - y_2 \\ y_1 + y_2 \end{bmatrix}$ we get $Q = 3y_1^2 + y_2^2$.

d) Sketch the graph of $Q(\vec{x}) = 1$ showing both the original and new axes.

Solution: Sketching we get



7. Show that if A is a symmetric matrix, then eigenvectors corresponding to distinct eigenvalues are orthogonal to each other.

Solution: Let $\vec{v}_1$, $\vec{v}_2$ be eigenvectors of A with corresponding eigenvalues λ_1 and λ_2 with $\lambda_1 \neq \lambda_2$. Then

$$A\vec{v}_1 \cdot \vec{v}_2 = (\lambda_1 \vec{v}_1) \cdot \vec{v}_2 = \lambda_1(\vec{v}_1 \cdot \vec{v}_2),$$

and

$$\vec{v}_1 \cdot A\vec{v}_2 = \vec{v}_1 \cdot \lambda_2 \vec{v}_2 = \lambda_2(\vec{v}_1 \cdot \vec{v}_2).$$

But, since A is symmetric we have $A\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot A\vec{v}_2$, thus

$$\lambda_1(\vec{v}_1 \cdot \vec{v}_2) = \lambda_2(\vec{v}_1 \cdot \vec{v}_2) \Rightarrow \vec{v}_1 \cdot \vec{v}_2 = 0,$$

since $\lambda_1 \neq \lambda_2$.

8. Let A be an $n \times n$ matrix with eigenvector $\vec{v}$ and let $\vec{u}$ be orthogonal to $\vec{v}$.

a) Prove that if A is symmetric, then $A\vec{u}$ is orthogonal to $\vec{v}$.

Solution: We have $\vec{v} \cdot A\vec{u} = A\vec{v} \cdot \vec{u}$ since A is symmetric. Hence

$$\vec{v} \cdot A\vec{u} = A\vec{v} \cdot \vec{u} = (\lambda \vec{v}) \cdot \vec{u} = \lambda(\vec{v} \cdot \vec{u}) = 0.$$

b) Give an example of a 2×2 matrix A with an eigenvector $\vec{v}$, and a vector $\vec{u}$ which is orthogonal to $\vec{v}$ such that $A\vec{u}$ is not orthogonal to $\vec{v}$.

Solution: One example is $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Observe that $\vec{u}$ is orthogonal to $\vec{v}$, but $A\vec{v} = \vec{v}$ and $A\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ which is not orthogonal to $A\vec{v}$.

9. Let V be an inner product space, and let $\vec{v}_1, \ldots, \vec{v}_k$ be (not necessarily linearly independent) vectors in V. Suppose that $S = \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$. For a vector $\vec{x} \in V$, prove that $\vec{x} \in S^\perp$ if and only if $\langle \vec{x}, \vec{v}_i \rangle = 0$ for each $i = 1, \ldots, k$.

Solution: ($\Rightarrow$) Let $\vec{x} \in S^\perp$. Then $\langle \vec{x}, \vec{y} \rangle = 0$ for every $y \in S$. In particular, $\vec{v}_1, \ldots, \vec{v}_k$ are in S, hence $\langle \vec{x}, \vec{v}_i \rangle = 0$ for each i.

($\Leftarrow$) Suppose that $\langle \vec{x}, \vec{v}_i \rangle = 0$ for each i. For every $y \in S$, $y = a_1\vec{v}_1 + \cdots + a_k\vec{v}_k$ for some a_i's. So

$$\begin{aligned} \langle \vec{x}, \vec{y} \rangle &= \langle \vec{x}, a_1\vec{v}_1 + \cdots + a_k\vec{v}_k \rangle \\ &= a_1 \langle \vec{x}, \vec{v}_1 \rangle + \cdots + a_k \langle \vec{x}, \vec{v}_k \rangle \\ &= 0. \end{aligned}$$

Since $\langle \vec{x}, \vec{y} \rangle = 0$ for every $\vec{y} \in S$, $\vec{x} \in S^\perp$.

10. Let A be an invertible symmetric matrix. Prove that if the quadratic form $\vec{x}^T A\vec{x}$ is positive definite, then so is the quadratic form $\vec{x}^T A^{-1}\vec{x}$.

Solution: If $\vec{x}^T A\vec{x}$ is positive definite, then the eigenvalues of A are all positive. Let λ be any eigenvalue of A and let $\vec{v}$ be the corresponding eigenvector. Then we have

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ A^{-1}A\vec{v} &= A^{-1}(\lambda\vec{v}) \\ \vec{v} &= \lambda A^{-1}\vec{v} \\ A^{-1}\vec{v} &= \frac{1}{\lambda}\vec{v} \end{aligned}$$

Hence, $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}. Thus, all the eigenvalues of A^{-1} are also positive, hence A^{-1} is also positive definite.