MATH 136 Midterm 1

(1) Let $\vec{x} = (2, -1, 4)$ and $\vec{y} = (-2, 0, 1)$

(a) Determine $-2\vec{y} - 2(\vec{x} - 3\vec{y})$

Soln: $-2(-2, 0, 1) - 2((2, -1, 4) - 3(-2, 0, 1))$

$= (-4, 0, -2) - 2(8, -1, 1)$

$= (-12, 2, -4)$

(b) Determine $\vec{z}$ such that $-3\vec{x} + \vec{z} = 2\vec{y}$

Soln : $-3(2, -1, 4) + (z_1, z_2, z_3) = 2(-2, 0, 1)$

$\Rightarrow (-6, 3, -12) + (z_1, z_2, z_3) = (-4, 0, 2)$

$\Rightarrow (z_1, z_2, z_3) = (-4, 0, 2) - (-6, 3, -12)$

$\Rightarrow \vec{z} = (z_1, z_2, z_3) = (2, -3, 14)$

(2) A is the point $(2, -1, 4)$ and B is $(-2, 0, 1)$. Determine the distance from A to B.

Soln : $\overrightarrow{AB} = (-2 - 2, 0 - (-1), 1 - 4) = (-4, 1, -3)$

$\|\overrightarrow{AB}\| = \sqrt{(-4)^2 + 1^2 + (-3)^2} = \sqrt{26}$

(3) Let $\vec{x} = (2, -1, 4)$ and $\vec{y} = (-2, 0, 1)$, determine $\vec{x} \cdot \vec{y}$.

Soln : $\vec{x} \cdot \vec{y} = 2 \cdot (-2) + (-1) \cdot 0 + 4 \cdot (1) = -4 + 0 + 4 = 0$

(4) Prove that $\|\vec{u}\| = \|\vec{v}\|$ **iff** $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = 0$

***Proof*:** $(\Rightarrow)$

$\|\vec{u}\| = \|\vec{v}\| \Rightarrow \|\vec{u}\|^2 = \|\vec{v}\|^2 \Rightarrow \|\vec{u}\|^2 - \|\vec{v}\| = 0$

$\Rightarrow \vec{u} \cdot \vec{u} - \vec{v} \cdot \vec{v} = 0$

$\Rightarrow \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{v} = 0$

$\Rightarrow \vec{u} \cdot (\vec{u} + \vec{v}) - \vec{v} \cdot \vec{u} - \vec{v} \cdot \vec{v} = 0$

$\Rightarrow \vec{u} \cdot (\vec{u} + \vec{v}) - \vec{v} \cdot (\vec{u} + \vec{v}) = 0$

$\Rightarrow (\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = 0$

$(\Leftarrow)$

$(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = 0$

$\Rightarrow \vec{u} \cdot \vec{u} + \vec{u} \cdot -\vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot -\vec{v} = 0$

$\Rightarrow \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{v} = 0$

$\Rightarrow \vec{u} \cdot \vec{u} - \vec{v} \cdot \vec{v} = 0$

$\Rightarrow \|\vec{u}\|^2 - \|\vec{v}\|^2 = 0$

$\Rightarrow \|\vec{u}\|^2 = \|\vec{v}\|^2$

$\Rightarrow \|\vec{u}\| = \|\vec{v}\|$

(5) Find an equation for the plane through $A(3,-1,2)$ and parallel to
$2\cdot x_1 - x_2 + 4x_3 = 4$

Soln : $\vec{n} = (2,-1,4)$

$\Rightarrow n_1x_1 + n_2x_2 + n_3x_3 = n_1a_1 + n_2a_2 + n_3a_3$

$\Rightarrow 2x_1 + (-1)x_2 + 4x_3 = 2a_1 + (-1)a_2 + 4a_3$

$\Rightarrow 2x_1 - x_2 + 4x_3 = 2(3) + (-1)(-1) + 4(2)$

$\Rightarrow 2x_1 - x_2 + 4x_3 = 15$

(6) Find the point of intersection for the given line and plane

Line : $\vec{x} = (-1,-2,4) + t(0,2,-1),\ t \in \mathbb{R}$

Plane : $3x_1 + x_2 - 4x_3 = 4$

Soln :

From the line we have,

$x_1 = -1$

$x_2 = -2 + 2\cdot t$

$x_3 = 4 - t$

Substituting this into the plane equation we get,

$3(-1) + (-2 + 2\cdot t) - 4(4 - t) = 4$

$\Rightarrow -3 - 2 + 2\cdot t - 16 + 4\cdot t = 4$

$\Rightarrow 6\cdot t = 25$

$\Rightarrow t = \frac{25}{6}$

Subsituting this back into our line equation,

$x_1 = -1$

$x_2 = -2 + 2\left(\frac{25}{6}\right) = \frac{19}{3}$

$x_3 = 4 - \frac{25}{6} = -\frac{1}{6}$

Thus the point of intersection is $\left(-1, \frac{19}{3}, -\frac{1}{6}\right)$

(7) Use projection to find the distance from the point $A(3,-1,2)$
to the plane
$2x_1 - x_2 + 4x_3 = 4.$

Soln : Consider the point $Q(1,-2,0)$ on the plane

$\overrightarrow{QA} = (2,1,2),\ \vec{n} = (2,-1,4)$

$$proj_{\vec{n}}\left(\overrightarrow{QA}\right) = \frac{\left(\overrightarrow{QA} \bullet \vec{n}\right)}{\|\vec{n}\|^2}\vec{n} = \frac{(4 - 1 + 8)}{4 + 1 + 16}(2,-1,4) = \frac{11}{21}(2,-1,4)$$

$$= \left(\frac{22}{21}, -\frac{11}{21}, \frac{44}{21}\right)$$

$$\left\| proj_{\vec{n}}\left(\overrightarrow{QA}\right) \right\| = \sqrt{\left(\frac{22}{21}\right)^2 + \left(-\frac{11}{21}\right)^2 + \left(\frac{44}{21}\right)^2} = \sqrt{\frac{121}{21}}$$

(8) Use projectionto find the distance from the point $A(3, 1)$ to the line $\underline{x} = (-1, 2) + t(3, 4)$.

Soln : Consider the point $Q(-1, 2)$ on the line

$\overrightarrow{QA} = (4,-1)$, $\vec{d} = (3, 4)$

$$proj_{\vec{d}}\left(\overrightarrow{QA}\right) = \frac{\left(\overrightarrow{QA} \cdot \vec{d}\right)}{\left\|\vec{d}\right\|^2}\vec{d} = \frac{(12 - 4)}{9 + 16}(3, 4) = \frac{8}{25}(3, 4) = \left(\frac{24}{25}, \frac{32}{25}\right)$$

$$perp_{\vec{d}}\left(\overrightarrow{QA}\right) = \overrightarrow{QA} - proj_{\vec{d}}\left(\overrightarrow{QA}\right) = (4,-1) - \left(\frac{24}{25}, \frac{32}{25}\right) = \left(\frac{76}{25}, \frac{-57}{25}\right)$$

$$\left\|perp_{\vec{d}}\left(\overrightarrow{QA}\right)\right\| = \sqrt{\left(\frac{76}{25}\right)^2 + \left(\frac{-57}{25}\right)^2} = \frac{19}{5}$$

(9) Let $\vec{x} = (2,-1, 4)$ and $\vec{y} = (-2, 0, 1)$, determine $\vec{x} \times \vec{y}$.

Soln : $\vec{x} \times \vec{y} = \left(x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1\right)$

$= ((-1)1 - 4(0), 4(-2) - 1(2), 2(0) - (-2)(-1))$

$= (-1,-10,-2)$

Determine if $\{(1,-1,3),(2,1,0),(3,0,4)\}$ is linearlydependent or independent

Soln : Write vectors in the form $c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} = \vec{0}$

$c_1(1,-1,3) + c_2(2,1,0) + c_3(3,0,4) = (0,0,0)$

Expandingwe get the system of linear equations:

$c_1 + 2c_2 + 3c_3 = 0$

$-1c_1 + c_2 = 0$

$3c_1 + 4c_3 = 0$

Writingthe system as a coefficient matrix we have :

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 3 & 0 & 4 \end{bmatrix} R2 : R2 + R1, R3 : R3 - 3\,R1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & -6 & -5 \end{bmatrix} R2 : \frac{R2}{3}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -6 & -5 \end{bmatrix} R1 : R1 - 2\,R2, R3 : R3 + 6\,R2$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} R1 : R1 - R3, R2 : R2 - R3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus $c_1 = c_2 = c_3 = 0$, and the vectors are lnearlyindependent

(10) Solve the following system :

$3x_1 + ix_2 + (2+i)x_3 = 3i$

$-ix_1 + x_2 + x_3 = 1$

$x_1 + x_2 + (2+i)x_3 = i$

Soln :

$$\left[\begin{array}{cccc|} 3 & i & (2+i) & 3i \\ -i & 1 & 1 & 1 \\ 1 & 1 & (2+i) & i \end{array}\right] R1 \leftrightarrow R2$$

$$\left[\begin{array}{cccc|} 1 & 1 & (2+i) & i \\ -i & 1 & 1 & 1 \\ 3 & i & (2+i) & 3i \end{array}\right] R3 : R3 - 3\,R1$$

$$\left[\begin{array}{cccc|} 1 & 1 & (2+i) & i \\ -i & 1 & 1 & 1 \\ 0 & i-3 & -4-2i & 0 \end{array}\right] R2 : R2 \cdot i$$

$$\left[\begin{array}{cccc|} 1 & 1 & (2+i) & i \\ 1 & i & i & i \\ 0 & i-3 & -4-2i & 0 \end{array}\right] R2 : R2 - R1$$

$$\left[\begin{array}{cccc|} 1 & 1 & (2+i) & i \\ 0 & i-1 & -2 & 0 \\ 0 & i-3 & -4-2i & 0 \end{array}\right] R2 : R2 - R3$$

$$\left[\begin{array}{cccc|} 1 & 1 & (2+i) & i \\ 0 & 2 & 2+2i & 0 \\ 0 & i-3 & -4-2i & 0 \end{array}\right] R2 : \frac{R2}{2}$$

$$\left[\begin{array}{cccc|} 1 & 1 & (2+i) & i \\ 0 & 1 & 1+i & 0 \\ 0 & i-3 & -4-2i & 0 \end{array}\right] R1 : R1 - R2, R3 : R3 + 3\,R2$$

$$\left[\begin{array}{cccc|} 1 & 0 & 1 & i \\ 0 & 1 & 1+i & 0 \\ 0 & i & -1+i & 0 \end{array}\right] R3 : R3 \cdot i$$

$$\left[\begin{array}{cccc|} 1 & 0 & 1 & i \\ 0 & 1 & 1+i & 0 \\ 0 & -1 & -i-1 & 0 \end{array}\right] R3 : R3 + R2$$

$$\left[\begin{array}{cccc|} 1 & 0 & 1 & i \\ 0 & 1 & 1+i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_3 = i \Rightarrow x_1 = i - x_3$

$x_2 + (1+i)x_3 = 0 \Rightarrow x_2 = -(1+i)x_3 = (-1-i)x_3$

x_3 is a free variable. Let $x_3 = t, t \in \mathbb{R}$

$(x_1, x_2, x_3) = (i, 0, 0) + t(-1, (-1-i), 1)$

True or False:

(1) If A is a 4×3 matrix and $\vec{b}$ is any vector in $\mathbb{R}^4$ such that $A\vec{x} = \vec{b}$ has a solution, then the solutionis unique.

(2) If a set of vectors contains the zero vector, the set is linearlydependent

(3) No vector in $\mathbb{R}^3$ is orthogonal to itself.

(4) All sets of 4 vectors in $\mathbb{R}^3$ are linearly dependent.

(5) If A and B are matrices such that $AB = 0$ then $A = 0$ or $B = 0$.

(1) T
(2) T
(3) F
(4) T
(5) F