Math 136 **Sample Term Test 2 # 2**

NOTES: - In addition to these questions you should also do questions 7, 10 b from sample term test 1 # 2.

1. Short Answer Problems

a) Let $S = \{v_1, \dots, v_n\}$ be a non-empty subset of a vector space V. Define the statement S is linearly independent.

b) Write the definition of a subspace S of a vector space V.

c) Write the definition of the dimension of a vector space V.

d) Prove that $0\mathbf{x} = \mathbf{0}$ for any $\mathbf{x} \in V$.

e) Is it true that if a set S with more than one vector is linearly dependent then every vector $v \in S$ can be written as a linear combination of the other vectors. Justify your answer.

2. Let $\beta = \{x^2 - 4x + 4, x - 2, 1\}$.

a) Show that $\text{span}(\beta) = P_2$.

b) Let $\mathbf{w} = x^2 + x + 1$. Find the β coordinate vector of $\mathbf{w}$.

3. Determine, with proof, which of the following are subspaces of the given vector space.

a) $S = \{ax^2 + bx + c \mid b^2 - 4ac \neq 0\}$ of P_2.

b) $D = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; ad - bc = 0 \right\}$ of $M(2, 2)$.

c) $A = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a + b + c + d = 0 \right\}$ of $M(2, 2)$.

4. Let $A = \begin{bmatrix} 3 & 6 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 3 \end{bmatrix}$ and let L be a linear mapping with matrix A.

a) Find a basis for the nullspace of L.

b) Find a basis for the range of L.

5. Find a basis for the following subspaces of $\mathbb{R}^3$ and state the dimension of the subspace.

a) $\text{span}\{(2, 4, 6), (4, 5, 6), (1, 1, 1)\}$

b) The plane $2x_1 - x_2 - x_3 = 0$.

6. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $S = \{v_1, \dots, v_p\}$ be a non-empty linearly independent subset of $\mathbb{R}^n$. Let $T(S) = \{T(v_1), \dots, T(v_p)\}$.

a) Prove that if the nullspace of T is $\{\vec{0}\}$ then $T(S)$ is also linearly independent.

b) Prove that if the nullspace of T is $\{\vec{0}\}$ and $n = m = p$ then $T(S)$ is a basis for $\mathbb{R}^n$.

c) Prove that if $n = m = p$ and $T(S)$ is linearly independent then the nullspace of T is $\{\vec{0}\}$.

7. Let $S = \{(a, b) \in \mathbb{R}^2 \mid b > 0\}$ and define addition by $(a, b) + (c, d) = (ad + bc, bd)$ and define scalar multiplication by $k(a, b) = (kab^{k-1}, b^k)$, then S is a vector space over $\mathbb{R}$.

a) Find the zero element for the vector space S.

b) Let $\mathbf{x} = (a, b) \in S$. Find the additive inverse of $\mathbf{x}$.

c) Prove that $k(\mathbf{x} + \mathbf{y}) = k\mathbf{x} + k\mathbf{y}$ for any $\mathbf{x}, \mathbf{y} \in S$ and any $k \in \mathbb{R}$.

d) Construct a basis for the vector space S over $\mathbb{R}$. Justify your answer.

Math 136 **Sample Term Test 2 # 2 Answers**

NOTE: - Only answers are provided here (and some proofs). On the test you **must** provide full and complete solutions to receive full marks.

1. Short Answer Problems

a) Let $S = \{v_1, \ldots, v_n\}$ be a non-empty subset of a vector space V. Define the statement S is linearly independent.

Solution: S is linearly independent if the only solution of $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$ is $c_1 = c_2 = \cdots = c_n = 0$.

b) Write the definition of a subspace S of a vector space V.

Solution: S is a subspace of V if S is a subset of V and S is a vector space using the same operations as V.

c) Write the definition of the dimension of a vector space V.

Solution: The dimension of V is the number of elements in any basis for V.

d) Prove that $0\mathbf{x} = \mathbf{0}$ for any $\mathbf{x} \in V$.

Solution: By vector space axioms 3,4,2,8 and 10, $0\mathbf{x} = 0\mathbf{x} + 0 = 0\mathbf{x} + (\mathbf{x} + (-\mathbf{x})) = (0\mathbf{x} + \mathbf{x}) + (-\mathbf{x}) = (0 + 1)\mathbf{x} + (-\mathbf{x}) = 1\mathbf{x} + (-\mathbf{x}) = \mathbf{x} + (-\mathbf{x}) = 0$.

e) Is it true that if a set S with more than one vector is linearly dependent then every vector $v \in S$ can be written as a linear combination of the other vectors. Justify your answer.

Solution: It is false.

2. Let $\beta = \{x^2 - 4x + 4, x - 2, 1\}$.

a) Show that $\text{span}(\beta) = P_2$.

Solution: Show the system $a(x^2 - 4x + 4) + b(x - 2) + c(1) = px^2 + qx + r$ is consistent for any p, q, r.

b) Let $\mathbf{w} = x^2 + x + 1$. Find the β coordinate vector of $\mathbf{w}$.

Solution: $(\mathbf{w})_\beta = (1, 5, 7)$

3. Determine, with proof, which of the following are subspaces of the given vector space.

a) $S = \{ax^2 + bx + c \mid b^2 - 4ac \neq 0\}$ of P_2.

Solution: S is not a subspace.

b) $D = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; ad - bc = 0 \right\}$ of $M(2, 2)$.

Solution: D is not a subspace.

c) $A = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a + b + c + d = 0 \right\}$ of $M(2, 2)$.

Solution: A is a subspace of $M(2, 2)$.

4. Let $A = \begin{bmatrix} 3 & 6 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 3 \end{bmatrix}$ and let L be a linear mapping with matrix A.

a) Find a basis for the nullspace of L.

Solution: A basis for the nullspace of L is $\{-2, 1, 0\}$.

b) Find a basis for the range of L.

Solution: $\{(3,1,2),(1,4,3)\}$ is a basis for the range of L.

5. Find a basis for the following subspaces of $\mathbb{R}^3$ and state the dimension of the subspace.

a) $\text{span}\{(2,4,6),(4,5,6),(1,1,1)\}$

Solution: A basis for $\text{span}\{(2,4,6),(4,5,6),(1,1,1)\}$ is $\{(2,4,6),(4,5,6)\}$ and hence the dimension is 2.

b) The plane $2x_1 - x_2 - x_3 = 0$.

Solution: The set $\{(1,0,2),(0,1,-1)\}$ is a basis.

6. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $S = \{v_1, \ldots, v_p\}$ be a non-empty linearly independent subset of $\mathbb{R}^n$. Let $T(S) = \{T(v_1), \ldots, T(v_p)\}$.

a) Prove that if the nullspace of T is $\{\vec{0}\}$ then $T(S)$ is also linearly independent.

Solution: If nullspace$(T) = \{\mathbf{0}\}$ and $a_1, a_2, ..., a_p$ are scalars such that

$$a_1T(v_1) + a_2T(v_2) + ... + a_pT(v_p) = \mathbf{0}$$

then by the linearity of T,

$$T(a_1v_1 + a_2v_2 + ... + a_pv_p) = \mathbf{0}$$

which implies that $a_1v_1 + a_2v_2 + ... + a_pv_p = \vec{0}$, since the nullspace is trivial.

Furthermore, $S = \{v_1, v_2, ..., v_p\}$ is linearly independent, this implies $a_1 = a_2 = ... = a_p = 0$, which shows that $T(S)$ is also linearly independent.

b) Prove that if the nullspace of T is $\{\vec{0}\}$ and $n = m = p$ then $T(S)$ is a basis for $\mathbb{R}^n$.

Solution: If nullspace$(T) = \{\mathbf{0}\}$, we showed that $T(S)$ is linearly independent. If $n = m = p$ also, then $T(S)$ is a linearly independent subset of $\mathbb{R}^n$ with n elements. By a theorem from text, $T(S)$ must be a basis for $\mathbb{R}^n$.

c) Prove that if $n = m = p$ and $T(S)$ is linearly independent then the nullspace of T is $\{\vec{0}\}$.

Solution: If $n = m = p$ and $T(S)$ is linearly independent, S is a linearly independent set of n vectors in $\mathbb{R}^n$ and is thus a basis.

If $T(v) = \mathbf{0}$, for some $v \in \mathbb{R}^n$, there exist $a_1, a_2, ..., a_n \in \mathbb{R}$ such that,

$$v = a_1v_1 + a_2v_2 + ... + a_nv_n$$

then $T(a_1v_1 + a_2v_2 + ... + a_nv_n) = \mathbf{0}$ and $a_1T(v_1) + a_2T(v_2) + ... + a_nT(v_n) = \mathbf{0}$ but $T(S) = \{T(v_1), T(v_2), ..., T(v_n)\}$ is linearly independent, this shows that $a_1 = a_2 = ... = a_n = 0$ and $v = 0$. $\Rightarrow$ nullspace$(T) = \{\mathbf{0}\}$

7. Let $S = \{(a,b) \in \mathbb{R}^2 \mid b > 0\}$ and define addition by $(a,b)+(c,d) = (ad+bc, bd)$ and define scalar multiplication by $k(a,b) = (kab^{k-1}, b^k)$, then S is a vector space over $\mathbb{R}$.

a) Find the zero element for the vector space S.

Solution: The zero element is $(0,1)$ since $(a,b)+(0,1) = (a(1)+b(0), b(1)) = (a,b)$.

b) Let $x = (a,b) \in S$. Find the additive inverse of x.

Solution: We need $(a,b)+(c,d) = (0,1)$, so we must have $d = \frac{1}{b}$ and hence we need $\frac{a}{b} + bc = 0$ so $c = -\frac{a}{b^2}$.

c) Prove that $k(\mathbf{x}+\mathbf{y}) = k\mathbf{x} + k\mathbf{y}$ for any $\mathbf{x}, \mathbf{y} \in S$ and any $k \in \mathbb{R}$.

Solution: Let $\mathbf{x} = (a,b)$ and $\mathbf{y} = (c,d)$ then we have

$$\begin{aligned}
k(\mathbf{x}+\mathbf{y}) &= k(ad+bc, bd) \\
&= (k(ad+bc)(bd)^{k-1}, (bd)^k) \\
&= (kad(bd)^{k-1} + kbc(bd)^{k-1}, b^k d^k) \\
&= (kcd^{k-1}b^k + kab^{k-1}d^k, b^k d^k) \\
&= (kab^{k-1}, b^k) + (kcd^{k-1}, d^k) \\
&= k(a,b) + k(c,d) = k\mathbf{x} + k\mathbf{y}
\end{aligned}$$

d) Construct a basis for the vector space S over $\mathbb{R}$. Justify your answer.

Solution: $\text{span}((1,1),(0,e)) = S$ is a basis.