Math 237 **F10 Midterm Solutions**

1. Short Answer Problems

[2] a) Let $f : \mathbb{R}^2 \to \mathbb{R}$. State the definition of $\lim_{(x,y)\to(a,b)} f(x, y) = L$.

Solution: For every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < \|(x, y) - (a, b)\| < \delta$.

[2] b) Let $f : \mathbb{R}^2 \to \mathbb{R}$. State the definition of the directional derivative of f in the direction of the unit vector $\hat{u} = (u_1, u_2)$ at the point (a, b).

Solution: $D_{\hat{u}} f(a, b) = \frac{d}{ds}\big[f(a + su_1, b + su_2)\big]\Big|_{s=0}$.

[2] c) Let $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f \in C^2$. State the definition of the second degree Taylor polynomial $P_{2,(a,b)}(x, y)$ of f at (a, b).

Solution: $P_{2,(a,b)}(x, y) = f(a, b)+f_x(a, b)(x-a)+f_y(a, b)(y-b)+\frac{1}{2}f_{xx}(a, b)(x-a)^2+f_{xy}(a, b)(x-a)(y - b) + \frac{1}{2}f_{yy}(a, b)(y - b)^2$

[2] d) Let $f : \mathbb{R}^3 \to \mathbb{R}$. What is the equation for the tangent plane to the level surface $f(x, y, z) = 3$ at (a, b, c)?

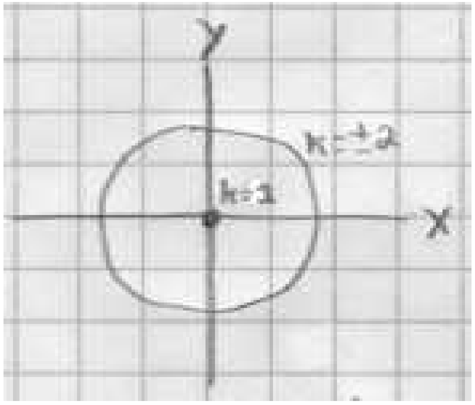
Solution: $\nabla f(a, b, c) \cdot (x - a, y - b, z - c) = 0$

[2] e) If f_x and f_y are both continuous at (a, b) what two things can we conclude about $f(x, y)$ at (a, b)?

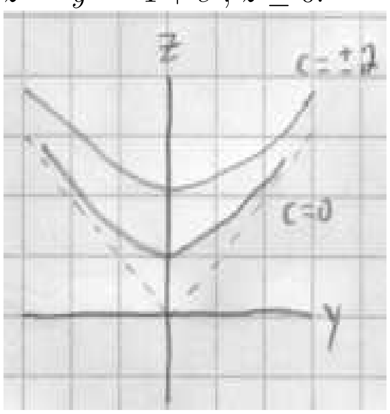
Solution: f is differentiable at (a, b) and f is continuous at (a, b).

[3] **2.** Let $f(x,y)=\sqrt{1+x^2+y^2}$. Sketch level curves and cross sections of $z=f(x,y)$.

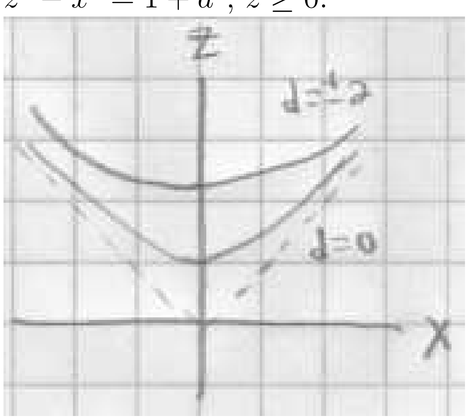
Solution: The level curves are $k=f(x,y)=\sqrt{1+x^2+y^2}$. Simplifying gives $k^2-1=x^2+y^2$.



The cross sections $x=c$ are $z=f(c,y)=\sqrt{1+c^2+y^2}$, $z>1$. Simplifying gives $z^2-y^2=1+c^2$, $z\geq 0$.



The cross sections $y=d$ are $z=f(x,d)=\sqrt{1+x^2+d^2}$, $z>1$. Simplifying gives $z^2-x^2=1+d^2$, $z\geq 0$.



3. For each of the following, evaluate the limit or show that it does not exist.

[2] a) $\lim_{(x,y)\to(-1,2)} \dfrac{2x^3y^2}{|x|+4|y|}$.

Solution: Since $f(x,y) = \frac{2x^3y^2}{|x|+4|y|}$ is continuous at $(-1,2)$ by the continuity theorems we get

$$\lim_{(x,y)\to(-1,2)} \frac{2x^3y^2}{|x|+4|y|} = \frac{2(-1)^3(2)^2}{|-1|+4|2|} = -\frac{8}{9}$$

[3] b) $\lim_{(x,y)\to(0,0)} \dfrac{xy-2|x|-4|y|}{|x|+2|y|}$.

Solution: Approaching along $y = mx$ gives

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{mx^2-2|x|-4|mx|}{|x|+2|mx|} &= \lim_{(x,y)\to(0,0)} \frac{m|x|-2-4|m|}{1+2|m|} \\ &= \frac{-2-4|m|}{1+2|m|} = -2\end{aligned}$$

We now use the Squeeze Theorem to prove that the limit is -2. We have

$$\begin{aligned}\left|\frac{xy-2|x|-4|y|}{|x|+2|y|} - (-2)\right| &= \left|\frac{xy-2|x|-4|y|+2|x|+4|y|}{|x|+2|y|}\right| = \frac{|xy|}{|x|+2|y|} \\ &\leq \frac{(|x|+2|y|)|y|}{|x|+2|y|} = |y|\end{aligned}$$

Since $\lim_{(x,y)\to(0,0)} |y| = 0$, we get that

$$\lim_{(x,y)\to(0,0)} \frac{xy-2|x|-4|y|}{|x|+2|y|} = -2$$

by the Squeeze Theorem.

4. The temperature in a region of space is given by $T(x, y, z) = e^{-2x}(1 + 2y)(\frac{1}{1+3z})$.
A fly moves along the path $(x, y, z) = (2t, \sin t, e^t - 1)$.

[4] a) Find $\frac{dT}{dt}$ at $t = 0$.

Solution: We have $T = T(x(t), y(t), z(t))$ and when $t = 0$, we get $(x(t), y(t), z(t)) = (0, 0, 0)$. We have

$$T_x = -2e^{-2x}(1 + 2y)\frac{1}{1 + 3z} \qquad x'(t) = 2$$
$$T_y = e^{-2x}(2)\frac{1}{1 + 3z} \qquad y'(t) = \cos t$$
$$T_z = e^{-2x}(1 + 2y)\frac{-3}{(1 + 3z)^2} \qquad z'(t) = e^t$$

Since T, $x(t)$, $y(t)$, and $z(t)$ are all differentiable the Chain Rule gives

$$\begin{aligned}\frac{dT}{dt}(0) &= T_x(0, 0, 0)x'(0) + T_y(0, 0, 0)y'(0) + T_z(0, 0, 0)z'(0) \\ &= (-2)(2) + (2)(1) + (-3)(1) = -5\end{aligned}$$

[3] b) Observe that the direction of the fly's path at $t = 0$ is $(2, 1, 1)$. Find the directional derivative of T in the direction of the fly's path at $t = 0$.

Solution: We have $\hat{u} = \frac{(2,1,1)}{\|(2,1,1)\|} = \frac{1}{\sqrt{6}}(2, 1, 1)$. Then, since T is differentiable we have

$$D_{\hat{u}}(0, 0, 0) = \nabla T(0, 0, 0) \cdot \hat{u} = (-2, 2, -3) \cdot \frac{1}{\sqrt{6}}(2, 1, 1) = -\frac{5}{\sqrt{6}}$$

[2] c) Explain the physical difference between a) and b).

Solution: $\frac{dT}{dt}$ is the rate of change of temperature as observed by the fly as it moves in time along the curve. It is calculating the change of temperature per second.

$D_{\hat{u}}T$ is the rate of change of temperature per metre along the fly's path.

5. Let $f(x,y) = \frac{1}{xy}$ for $x > 0$ and $y > 0$.

[3] a) Find the linear approximation $L_{(1,1)}(x,y)$ of f at $(1,1)$.

Solution: We have $f_x = -\frac{1}{yx^2}$ and $f_y = -\frac{1}{xy^2}$ So,

$$\begin{aligned} L_{(1,1)}(x,y) &= f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) \\ &= 1 + (-1)(x-1) + (-1)(y-1) \\ &= 1 - (x-1) - (y-1) \end{aligned}$$

[3] b) Use Taylor's Theorem to show that if $x > 1$ and $y > 1$, then

$$|f(x,y) - L_{(1,1)}(x,y)| \leq \frac{3}{2}[(x-1)^2 + (y-1)^2].$$

Solution: We have

$$f_{xx} = \frac{2}{yx^3} \qquad f_{xy} = \frac{1}{x^2y^2} \qquad f_{yy} = \frac{2}{xy^3}$$

So, $f \in C^2$ for $x > 1$ and $y > 1$. Thus, by Taylor's Theorem, there exists a point $\underline{c} = (c,d)$ on the line segment between (x,y) and (a,b) such that

$$\begin{aligned} |f(x,y) - L_{(1,1)}(x,y)| &= \left| \frac{1}{2} f_{xx}(\underline{c})(x-1)^2 + f_{xy}(\underline{c})(x-1)(y-1) + \frac{1}{2} f_{yy}(y-1)^2 \right| \\ &\leq \frac{1}{2}|f_{xx}(\underline{c})|(x-1)^2 + |f_{xy}(\underline{c})||x-1||y-1| + \frac{1}{2}|f_{yy}(\underline{c})|(y-1)^2 \end{aligned}$$

For $x > 1$ and $y > 1$ we have $c > 1$ and $d > 1$, hence

$$|f_{xx}(\underline{c})| \leq 2 \qquad |f_{xy}(\underline{c})| \leq 1 \qquad |f_{yy}(\underline{c})| \leq 2$$

So,

$$\begin{aligned} |f(x,y) - L_{(1,1)}(x,y)| &\leq \frac{1}{2}(2)(x-1)^2 + |x-1||y-1| + \frac{1}{2}(2)(y-1)^2 \\ &= (x-1)^2 + \frac{1}{2}(x-1)^2 + \frac{1}{2}(y-1)^2 + (y-1)^2 \\ &= \frac{3}{2}\left[(x-1)^2 + (y-1)^2\right] \end{aligned}$$

6. Let $f(x,y) = \begin{cases} \frac{xy^2+y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

[2] a) Prove that f is continuous at $(0,0)$.

Solution: To prove that $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$ we use the Squeeze Theorem.

$$\left|\frac{xy^2+y^3}{x^2+y^2} - 0\right| \leq \frac{|x|(x^2+y^2)+|y|(x^2+y^2)}{x^2+y^2} = |x|+|y|$$

Since $\lim_{(x,y)\to(0,0)} |x|+|y| = 0$, we get that $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$ as required.

[6] b) Determine all points where f is differentiable.

Solution: We have

$$f_x = \frac{y^2(x^2+y^2)-(xy^2+y^3)(2x)}{(x^2+y^2)^2}$$
$$f_y = \frac{(2xy+3y^2)(x^2+y^2)-(xy^2+y^3)(2y)}{(x^2+y^2)^2}$$

By the continuity theorems f_x and f_y are both continuous for all $(x,y) \neq (0,0)$, hence f is differentiable for all $(x,y) \neq (0,0)$.

At $(0,0)$ we must use the definition of differentiability. We have

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$
$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{h^3}{h^3} = 1$$

Thus, $R_{1,(0,0)}(x,y) = f(x,y) - (0+y) = \frac{xy^2-x^2y}{x^2+y^2}$. Consider $\lim_{(x,y)\to(0,0)} \frac{|R_{1,(0,0)}(x,y)|}{\|(x,y)-(0,0)\|} = \frac{|xy^2-x^2y|}{(x^2+y^2)^{3/2}}$

Approaching the limit along $y = 2x$ gives

$$\lim_{x\to 0} \frac{|4x^3-2x^3|}{(x^2+4x^2)^{3/2}} = \lim_{x\to 0} \frac{2}{(5)^{3/2}} = \frac{2}{(5)^{3/2}} \neq 0$$

Hence, f is not differentiable at $(0,0)$.

Therefore f is differentiable on all of $\mathbb{R}^2$ except $(x,y) = (0,0)$.

[5] **7.** The Laplace equation for the scalar field $f(x,y)$ is $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0.$

Let $x = r\cos\theta$ and $y = r\sin\theta$. Show that

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Solution: By the Chain Rule we get

$$\begin{aligned}
f_r =& f_x\cos\theta + f_y\sin\theta \\
f_{rr} =& \big(f_{xx}\cos\theta + f_{yx}\sin\theta\big)\cos\theta + \big(f_{xy}\cos\theta + f_{yy}\sin\theta\big)\sin\theta \\
f_\theta =& -f_x r\sin\theta + f_y r\cos\theta \\
f_{\theta\theta} =& \big(-f_{xx} r\sin\theta + f_{yx} r\cos\theta\big)(-r\sin\theta) + \big(-f_{xy} r\sin\theta + f_{yy} r\cos\theta)(r\cos\theta) \\
& - f_x r\cos\theta - f_y r\sin\theta
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} =& f_{xx}\cos^2\theta + f_{yx}\cos\theta\sin\theta + f_{xy}\cos\theta\sin\theta + f_{yy}\sin^2\theta + \frac{1}{r}f_x\cos\theta \\
& + \frac{1}{r}f_y\sin\theta + f_{xx}\sin^2\theta - f_{yx}\cos\theta\sin\theta - f_{xy}\cos\theta\sin\theta + f_{yy}\cos^2\theta \\
& - \frac{1}{r}f_x\cos\theta - \frac{1}{r}f_y\sin\theta \\
=& f_{xx}(\cos^2\theta + \sin^2\theta) + f_{yy}(\cos^2\theta + \sin^2\theta) \\
=& f_{xx} + f_{yy}
\end{aligned}$$

[2] **8.** Let $f : \mathbb{R}^2 \to \mathbb{R}$. Prove that if f is differentiable at (a, b), then f is continuous at (a, b).

Solution: The error $R_{1,\underline{a}}(\underline{x})$ is defined by

$$R_{1,\underline{a}}(\underline{x}) = f(\underline{x}) - L_{\underline{a}}(\underline{x}).$$

On using the definition of $L_{\underline{a}}(\underline{x})$, this equation can be rearranged to read

$$f(\underline{x}) = f(\underline{a}) + \nabla f(\underline{a}) \cdot (\underline{x} - \underline{a}) + R_{1,\underline{a}}(\underline{x}). \tag{1}$$

We can write

$$R_{1,\underline{a}}(\underline{x}) = \frac{R_{1,\underline{a}}(\underline{x})}{\|\underline{x} - \underline{a}\|}\|\underline{x} - \underline{a}\|, \quad \text{for} \quad \underline{x} \neq \underline{a}.$$

Since f is differentiable and by the limit theorems, we get

$$\lim_{\underline{x} \to \underline{a}} R_{1,\underline{a}}(\underline{x}) = \lim_{\underline{x} \to \underline{a}} \frac{R_{1,\underline{a}}(\underline{x})}{\|\underline{x} - \underline{a}\|}\|\underline{x} - \underline{a}\| = 0 \cdot 0 = 0.$$

It now follows that

$$\lim_{\underline{x} \to \underline{a}} f(\underline{x}) = f(\underline{a}) + 0 + 0 = f(\underline{a}),$$

and so by definition, f is continuous at $\underline{a}$.

[2] **9.** Let $f(x, y) = x + y - \ln x - \ln y$ for $x > 0$, $y > 0$. Show that $f(\underline{x}) > L_{\underline{a}}(\underline{x})$ for all $\underline{x} \neq \underline{a} = (a, b)$ where $a > 0$ and $b > 0$.

Solution: We have $f_x = 1 - \frac{1}{x}$, $f_y = 1 - \frac{1}{y}$, $f_{xx} = \frac{1}{x^2}$, $f_{xy} = 0$, and $f_{yy} = \frac{1}{y^2}$. Since $f \in C^2$ for $x > 0$ and $y > 0$, we get by Taylor's Theorem that there exists a point $\underline{c} = (c, d)$ on the line segment joining (x, y) to (a, b) such that

$$\begin{aligned} f(x, y) - L_{\underline{a}}(\underline{x}) &= \frac{1}{2} f_{xx}(\underline{c})(x - a)^2 + f_{xy}(\underline{c})(x - a)(y - b) + \frac{1}{2} f_{yy}(\underline{c})(y - b)^2 \\ &= \frac{1}{2c^2}(x - a)^2 + \frac{1}{2d^2}(y - b)^2 > 0 \quad \text{ for } \underline{x} \neq \underline{a} \end{aligned}$$

Hence, $f(\underline{x}) > L_{\underline{a}}(\underline{x})$.