1. Short Answer Problems

a) Let $f : \mathbb{R}^2 \to \mathbb{R}$. What is the definition of f being continuous at a point (a, b)?

b) Let $f : \mathbb{R}^2 \to \mathbb{R}$. What condition on f_x and f_y guarantees that the linear approximation of f is a good approximation?

c) State Taylor's Theorem with second degree remainder.

d) Let $f : \mathbb{R}^3 \to \mathbb{R}$. What is the formula for the linear approximation of f at the point (a, b, c)?

2. Let $f(x, y) = \sqrt{|1 - x^2 - y^2|}$.

a) What is the domain and range of f?

b) Sketch the level curves and cross sections of $z = f(x, y)$.

3. Prove that if $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a, b) then f is continuous at (a, b).

4. Let $g : \mathbb{R}^2 \to \mathbb{R}$ and let $f(x, y) = g(y^2, xy)$. Find $\frac{\partial^2 f}{\partial x \partial y}$. What assumptions do you need to make about g so that you can apply the chain rule?

5. Let $f(x, y) = \ln(2x - 3y)$.

a) Find the linear approximation $L_{(2,1)}(x, y)$ of f.

b) Use Taylor's Theorem to show that $|R_{1,(2,1)}(x, y)| \leq \frac{15}{2}\left[(x-2)^2 + (y-1)^2\right]$ for $x \geq 2$, $y \leq 1$.

6. Let $f(x, y) = 2x^2 + xy^3$.

a) State the definition of the directional derivative $D_{\hat{u}} f(a, b)$ of f at (a, b) in the direction of the unit vector $\hat{u}$.

b) Find the rate of change of f at the point $(1, 2)$ in the direction of the vector $(1, -3)$.

c) In what direction from $(-2, 1)$ does f change most rapidly and what is the maximum rate of change.

7. Determine if each of the following limits exist. Evaluate the limits that exist.

a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - xy - y^2}{x^2 + y^2}$.

b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - |x| - |y|}{|x| + |y|}$.

8. Consider the function

$$f(x, y) = \begin{cases} \frac{x^{4/3}\, y}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

a) Where is f differentiable on its domain?

b) Based on your answer in part a), what can you conclude about the continuity of both f_x and f_y at $(0, 0)$?