1. Short Answer Problems

a) Let $f : \mathbb{R}^2 \to \mathbb{R}$. State the precise definition of $\lim_{(x,y)\to(a,b)} f(x, y) = L$.

b) Let $f : \mathbb{R}^2 \to \mathbb{R}$. If f_x and f_y are both continuous at (a, b), then what are two things you can say about f at (a, b)?

c) Let $f(x, y) = x^2 + xy + y^3$. What is the greatest rate of change of f at $(-1, 1)$?

d) If $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous second partial derivatives and $Hf(x, y) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for all points (x, y) in a neighborhood of (a, b) then what can you conclude about the accuracy of $L_{(a,b)}(x, y)$? Justify your answer.

2. Let $f(x, y) = \sqrt{y^2 - x^2}$.

a) Sketch the domain and state the range of f?

b) Sketch the level curves and cross sections of $z = f(x, y)$.

3. Prove that if $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a, b), then

$$D_{\mathbf{u}} f(a, b) = \nabla f(a, b) \cdot \mathbf{u}.$$

4. Let $g : \mathbb{R}^2 \to \mathbb{R}$ and let $f(x, y) = g(u, v)$ where $u = u(x, y) = xe^{xy}$ and $v = v(x, y) = x + y^2$. Find $\frac{\partial^2 f}{\partial x \partial y}$. What assumptions do you need to make about g so that you can apply the chain rule?

5. Use the second degree Taylor polynomial to approximate $(-0.99)^3(1.02)^2$.

6. Find the equation of the tangent plane to the surface $\sin z = \ln(x^2 + y)$ at $(2, -3, \pi)$.

7. Let $f(x, y) = \sqrt{(x^2 + y^2)}$. Use Taylor's Theorem to show that for $x \geq 1$, $y \geq 1$

$$|R_{1,(1,1)}(x, y)| \leq 2^{-3/2}\left[(x-1)^2 + (y-1)^2\right]$$

8. Determine if each of the following limits exist. Evaluate the limits that exist.

a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^4}{x^4 + y^8}$.

b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{-2x^2 + x^2y^2 - 2y^2}{x^2 + y^2}$.

9. Let $f(x, y) = \begin{cases} \frac{x^{7/3}y^{2/3}}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$

a) Determine if $f(x, y)$ is continuous at $(0, 0)$.

b) Determine all points where f is differentiable.

c) Based on your answer in part b), what can you conclude about the continuity of both f_x and f_y at $(0, 0)$?

d) Find the directional derivative of f at $(0, 0)$ in the direction of the vector $(1, 1)$.