University of Waterloo
Waterloo, Ontario

Mathematics 237
Mid-Term Test – Winter 2006

Time: 1 $\frac{1}{2}$ hours

Date: February 14, 2006

NO AIDS PERMITTED

Family Name: [redacted] Initials: [redacted] I.D. Number: [redacted]

Signature: [redacted]

Instructors:				
	☐	X. Liu	Section 001,	8:30 a.m.
	☐	D. Wolczuk	Section 002,	11:30 a.m.
	☑	B. Bodmann	Section 003,	11:30 a.m.

Your grade will be influenced by how clearly you express your ideas, and how well you organize your solutions.

Instructions:

1. Complete the information section above, indicating your instructor's name by a checkmark in the appropriate box.
2. Answer all questions. Two marks of each question are for exposition.
3. This examination has 8 pages.

FOR EXAMINERS' USE ONLY		
Questions	Maximum	Mark
1	18	18
2	12	12
3	14	14
4	14	14
5	27	26
6	15	10
Total	100	94

explanations

[18] 1. A function f is defined by

$$f(x,y) = (x-2y)^2$$

$\tan 2\theta = \frac{2a_{12}}{a_{22}-a_{11}} =$

a) Sketch the level curves for $C = 0, 1, 2$.

$f(x,y) = x^2 - 4xy + 4y^2$ ... parabolic cylinder, rotated

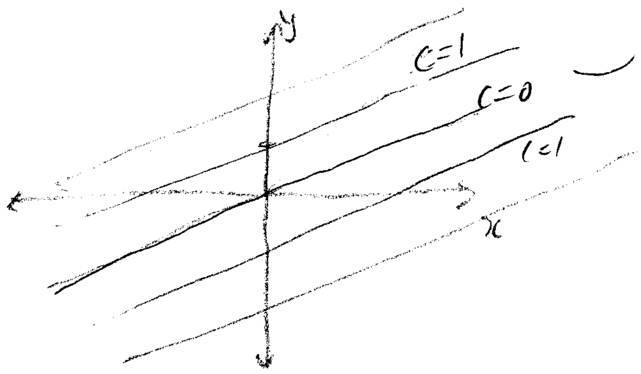
$C=0$: $0 = (x-2y)^2$: $x = 2y$, $x, y = 0$

$C=1$: $1 = (x-2y)^2$:

$1 = \pm(x-2y)$

$1 = x-2y$ $y = \frac{1}{2} - x$

$1 = 2y - x$ $y = \frac{1}{2} + x$



$C=2$

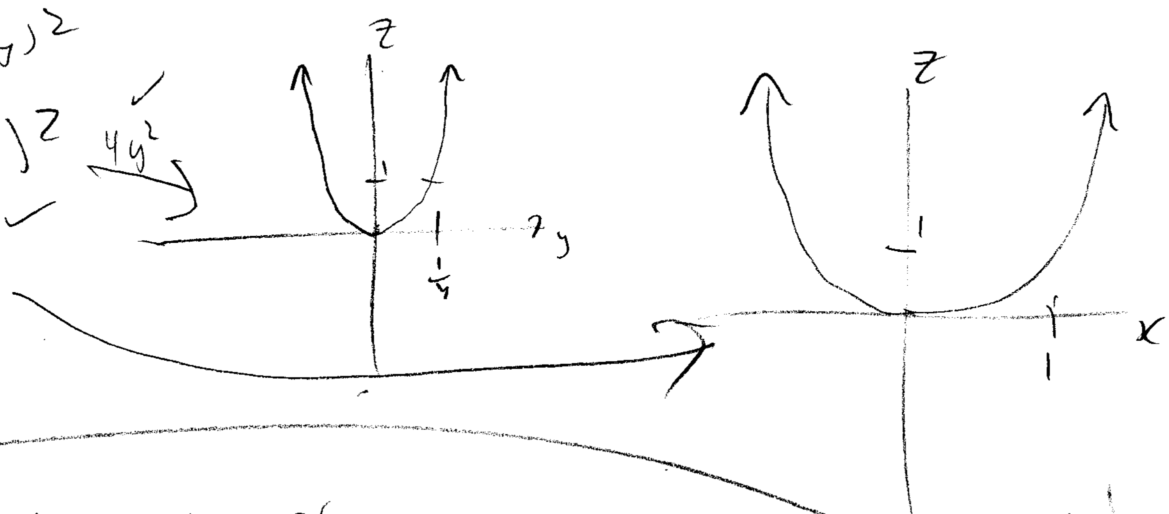
$\frac{6}{6}$

b) Sketch the cross-sections with the $x=0$ and $y=0$ planes.

$z = (x-2y)^2$

$z = (-2y)^2$ $4y^2$

$\frac{6}{6}$ $z = x^2$



$x^2 - 4yx + 4y^2$ $\frac{\partial f}{\partial x} = 2x - 4y$ $\frac{\partial f}{\partial y} = 8y - 4x$

c) Determine the equation of the tangent plane to the surface $z = f(x,y)$ at the point $(1,1,1)$.

$\frac{6}{6}$

$z = f(a,b) + \frac{\partial f}{\partial x}(x-a) + \frac{\partial f}{\partial y}(y-b)$ $(1,1,1)$

$z = 1 + (-2)(x-1) + (4)(y-1)$

$z = 1 - 2(x-1) + 4(y-1)$

[12] 2. The volume of a silo is given by

$$V(r,h) = \frac{2}{3}\pi r^3 + \pi r^2 h$$

where r is the radius and h is the height of its cylindrical wall.

a) Suppose the silo has a radius 5 m and a height 25 m. Estimate by linear approximation the change ΔV in volume if the radius is decreased by 0.1 m and the height is increased by 0.1 m.

$\Delta V = \frac{\partial V}{\partial r}\Delta r + \frac{\partial V}{\partial h}\Delta h$ $\quad \frac{\partial V}{\partial r} = \frac{2}{3}\pi 3r^2 + \pi h 2r$ ✓

$\Delta V = \left[(2\pi)(5m)^2 + \pi(25m)2(5m)\right](-0.1)$ $\quad \frac{\partial V}{\partial h} = \pi r^2$ ✓

$+\left[\pi(5m)^2\right](0.1m)$

$\Delta V = [50\pi + 250\pi]((-0.1)) + 25\pi(0.1)m$ ✓

$\Delta V = -\frac{1}{10}300\pi + 2.5\pi =$

$\Delta V = -30\pi + \frac{5}{2}\pi$ ✓

8/8

b) Discuss the validity of your estimation, i.e. explain why the approximation error is expected to be small.

$V(r,h)$ is differentiable at ✓ $(5m, 25m)$. (to see this note that the partial derivatives $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial h}$ exist and are continuous at $(5, 25)$ ✓.

By definition this means that the error in the linear approximation $f(x,y) - L(x,y)$ goes to zero faster than than the displacement of $\vec{x}$ from $\vec{a}$ as $\vec{x}$ goes to $\vec{a}$, and that the approximation is a good one. ✓

4/4

- Unique ...

[14] 3. A function is defined by

$$f(x,y) = e^x \sin y + 2x^2 y$$

a) Find the directional derivative of $f(x,y)$ at the point $\left(0, \frac{\pi}{2}\right)$ in the direction of the vector (3,4).

TGTDT

$D_{\vec{u}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u}$

$f_x = e^x \sin y + 4xy$; $f_y = e^x \cos y + 2x^2$

$\nabla f(0, \frac{\pi}{2}) = (1+0, 0+0) = (1, 0)$

$\vec{u} = \frac{(3,4)}{\|(3,4)\|} = \frac{(3,4)}{\sqrt{3^2+4^2}} = \left(\frac{3}{5}, \frac{4}{5}\right)$

$D_{\vec{u}} f(0, \frac{\pi}{2}) = (1,0) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) = \frac{3}{5} + 0 = \frac{3}{5}$

$\frac{8}{8}$

b) Find the direction and the rate of the most rapid change of f at $\left(0, \frac{\pi}{2}\right)$.

* Direction of most rapid increase is the direction of the gradient

* magnitude " " " magnitude "

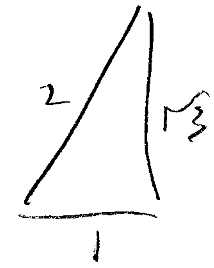
$\|\nabla f(0, \pi/2)\| = \|(1, 0)\| = \sqrt{1^2 + 0^2} = 1$

$\frac{6}{6}$ Then the gradient itself is a unit vector, and so it gives its normalized direction.

$\vec{u} = (1, 0)$.

[14] 4. Suppose $f(x, y)$ is differentiable, and $T(r, \theta) = f(r\cos\theta,\ r\sin\theta)$.

a) Calculate $\frac{\partial T}{\partial \theta}\left(2, \frac{\pi}{3}\right)$ if $\nabla f(1, \sqrt{3}) = (-1, 1)$.



$$\frac{\partial T}{\partial \theta} = D_1 f\, r(-\sin\theta) + D_2 f\, r\cos\theta$$

$$\frac{\partial T}{\partial \theta}\left(2, \frac{\pi}{3}\right) = -(-1)(2)\sin\left(\frac{\pi}{3}\right) + (1)(2)\cos\frac{\pi}{3} = 2\sin\frac{\pi}{3} + 2\cos\frac{\pi}{3}$$

$$= 2 \cdot \frac{1}{2} + 2\frac{\sqrt{3}}{2} = 1 + \sqrt{3}$$

because $\nabla f(1, \sqrt{3}) = (D_1 f, D_2 f) = (-1, 1)$

(b) Show $\left(\frac{\partial T}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial T}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$

$$\frac{\partial T}{\partial r} = D_1 f\cos\theta + D_2 f\sin\theta$$

$$\frac{\partial T}{\partial \theta} = -D_1 f\, r\sin\theta + D_2 f\, r\cos\theta$$

$$\left(\frac{\partial T}{\partial r}\right)^2 = (D_1 f\cos\theta)^2 + 2D_1 f D_2 f\cos\theta\sin\theta + (D_2 f\sin\theta)^2$$

$$\left(\frac{\partial T}{\partial \theta}\right)^2 = (D_1 f\, r\sin\theta)^2 - 2D_1 f D_2 f\, r^2\sin\theta\cos\theta + (D_2 f\, r\cos\theta)^2$$

$$\left(\frac{\partial T}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial T}{\partial \theta}\right)^2 = (D_1 f)^2\cos^2\theta + \cancel{2D_1 f D_2 f\cos\theta\sin\theta} + (D_2 f)^2\sin^2\theta$$

cancel

$$+ \frac{1}{\cancel{r^2}}\left((D_1 f)^2 \cancel{r^2}\sin^2\theta - \cancel{2D_1 f D_2 f\, r^2\sin\theta\cos\theta} + (D_2 f)^2\cancel{r^2}\cos^2\theta\right)$$

$$= (D_1 f)^2\cos^2\theta + (D_2 f)^2\sin^2\theta + (D_1 f)^2\sin^2\theta + (D_2 f)^2\cos^2\theta$$

$$= (D_1 f)^2(\cos^2\theta + \sin^2\theta) + (D_2 f)^2(\cos^2\theta + \sin^2\theta)$$

$$= (D_1 f)^2 + (D_2 f)^2$$

* which proves the above equation is true.

[27] 5. Let f be defined by

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

a) State the definition of differentiability of a function $f : R^2 \to R$ at a given point $\mathbf{a} = (a,b)$.

a function $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable at the point (a,b) iff

a) D_1f and D_2f exist

b) $$\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - L_{(a,b)}(x,y)|}{\|(x,y)-(a,b)\|} = 0$$

b) Show that the given function f is continuous at (0,0).

we must show that:

a) $f(0,0)$, $\lim_{(x,y)\to(a,a)} f(x,y)$ exist

b) $f(0,0) = \lim_{(x,y)\to(0,0)} f(x,y)$

let $\vec{x} = (x,y)$, $\vec{a} = (0,0)$

$\lim_{\vec{x}\to\vec{a}} \frac{x^2y}{x^2+y^2}$ limit probably exists, probably equals $\cancel{0}$ ($f(x,y) \sim \frac{u^3}{u^2}$)

prove that $|f(x,y) - 0| \le M(x,y)$ for some function $M(x,y)$ in order to show that $\lim_{\vec{x}\to\vec{a}} f(x,y) = \lim_{\vec{x}\to\vec{a}} M(x,y)$ squeeze thm

$$\left|\frac{x^2y}{x^2+y^2}\right| \le \frac{(x^2+y^2)|y|}{x^2+y^2} = |y|$$

-1

because $\lim_{\vec{x}\to\vec{a}} |y| = 0$, $\lim_{\vec{x}\to\vec{a}} f(x,y) = 0 = f(0,0)$, so $f(x,y)$ is continuous at $(0,0)$

5. c) For the given function f, calculate $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.

$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \quad h^2 0$

BAH! Clearly the top term in $f(0,0)$, $f(h,0)$, and $f(0,h)$ will all be $\emptyset$ and so $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ will both be $\emptyset$ at $(0,0)$. This is because the cross sections in either case are at the x,y axes and the value of f everywhere there is $\emptyset$.

d) Is f differentiable at $(0,0)$?

$\lim_{(x,y)\to(0,0)} \frac{|f(x,y) - L(x,y)|}{\|(x,y)-(0,0)\|}$ must equal $\emptyset$ too (already proved the other condition, above).

$L_a(x,y) = f(0,0) + 0(x-a) + 0(y-b) = 0$

$g(x,y) = \frac{|f(x,y)|}{\sqrt{x^2+y^2}} = \left|\frac{|x^2 y|}{(x^2+y^2)\sqrt{x^2+y^2}}\right| = \left|\frac{x^2|y|}{(x^2+y^2)^{3/2}}\right|$

$\lim_{\vec{x}\to\vec{a}} g(x,y)$

let $y = mx$

$\left|\frac{x^2|mx|}{(x^2+m^2x^2)^{3/2}}\right| = \left|\frac{|m||x^3|}{(x^2(1+m^2))^{3/2}}\right| = \left|\frac{|m|\cancel{|x^3|}}{(1+m^2)^{3/2}\cancel{|x^3|}}\right|$

because $\lim_{\vec{x}\to\vec{a}} g(x,y)$ depends on m, it is not unique, and therefore doesn't exist.

(10) [15] 6. Answer true (T) or false (F) for the following statements. You don't have to justify your answer, but you'll get a penalty of a negative mark of 2 for each wrong answer and zero marks for no answer.

[F] a) If $\frac{\partial f}{\partial x}(a,b)$ and $\frac{\partial f}{\partial y}(a,b)$ exist, then f is continuous at (a,b).

[T] b) If $f : R^2 \to R$ and $\nabla f(a,b) = (0,0)$ for all $(a,b) \in R^2$, then $f(x,y)$ is a constant.

[T] c) If both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous ~~at~~ (a,b), then f is differentiable at (a,b).

[F] d) $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if and only if $f(x,y)$ tends to L when (x,y) approaches (a,b) along any straight line. level curve

[X] e) The plane curve $\mathbf{F}(t) = \left(\frac{t^2}{2}+t-3, \ \frac{t^2}{2}-t+5\right)$ intersects the level curve of $f(x,y) = 5x^2 + 3y^2$ orthogonally when $t = 0$.

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