1. Short Answer Problems

[1] a) Let $f : \mathbb{R}^2 \to \mathbb{R}$. Write the precise definition of f being differentiable at (a, b).

If f_x and f_y both exist at (a, b) and $\lim\limits_{(x,y)\to(a,b)} \frac{|R_{1,(a,b)}(x,y)|}{\|(x,y)-(a,b)\|} = 0$, where $R_{1,(a,b)}(x, y) = f(x, y) - L_{(a,b)}(x, y)$, then f is differentiable at (a, b).

[2] b) If f_x and f_y are both continuous at (a, b) what two things can we conclude about $f(x, y)$?

f is differentiable at (a, b) and f is continuous at (a, b).

[1] c) What is the equation for the tangent plane to the surface $0 = f(x, y, z)$ at (a, b, c)?

$$\nabla f(a, b, c) \cdot (x - a, y - b, z - c) = 0.$$

[1] d) State the definition of the directional derivative $D_{\mathbf{u}} f(a, b)$ at a point (a, b) in the direction of a unit vector $\mathbf{u}$.

$$D_{\mathbf{u}} f(a, b) = \frac{d}{ds} f\big((a, b) + s(u_1, u_2)\big)\Big|_{s=0}.$$

OR

$$D_{\mathbf{u}} f(a, b) = \frac{d}{ds} f\big(\underline{a} + s\underline{u}\big)\Big|_{s=0}.$$

OR

$$D_{\mathbf{u}} f(a, b) = \lim_{h\to 0} \frac{f(\underline{a} + h\underline{u}) - f(\underline{a})}{h}.$$

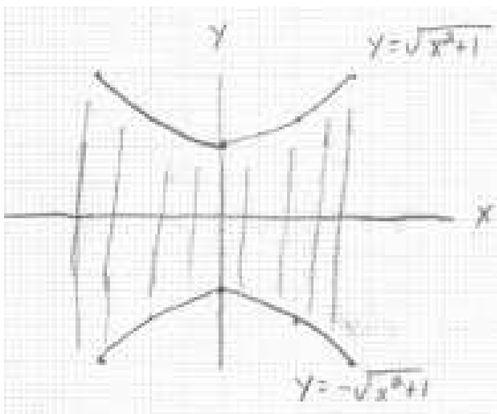
2. Let $f(x, y) = \sqrt{1 + x^2 - y^2}$.

[2] a) Sketch the domain of f. What is the range of f?

Solution: We must have $1 + x^2 - y^2 \geq 0$ hence we must have

$$y^2 \leq 1 + x^2 \Rightarrow |y| \leq \sqrt{1 + x^2} \Rightarrow -\sqrt{1 + x^2} \leq y \leq \sqrt{1 + x^2}.$$

Which gives the diagram below.



The range is clealy $z \geq 0$.

[3] b) Sketch the level curve $1 = f(x, y)$, the cross-section $x = 1$ and the cross section $y = 1$ of the surface $z = f(x, y)$.

Solution:

Level Curves: $z = 1$

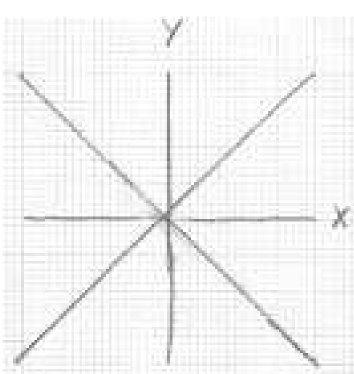
$$1 = \sqrt{1 + x^2 - y^2}$$
$$1 = 1 + x^2 - y^2$$
$$x^2 = y^2$$

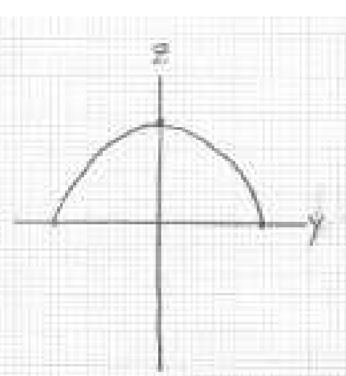
Cross sections: $x = 1$

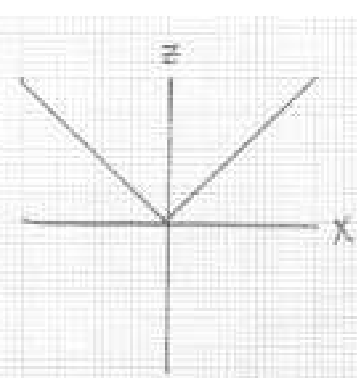
$$z = \sqrt{1 + 1^2 - y^2}$$
$$z = \sqrt{2 - y^2}$$

Cross sections: $y = 1$

$$z = \sqrt{1 + x^2 - 1^2}$$
$$z = \sqrt{x^2} = |x|$$







3. Determine if each of the following limits exist. Evaluate the limits that exist.

[2] a) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{(x^2+xy+y^2+1)\sin(x^2+y^2)}{x^2+y^2}$.

Solution: Since $(1,0)$ is a typical point we have

$$\lim_{(x,y)\to(1,0)} \frac{(x^2+xy+y^2+1)\sin(x^2+y^2)}{x^2+y^2} = 2\sin 1.$$

[3] b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sqrt[3]{x}y^2}{x+y^3}$.

Solution: Approaching the limit along $x = m^3y^3$ we get

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{\sqrt[3]{x}y^2}{x+y^3} &= \lim_{(x,y)\to(0,0)} \frac{\sqrt[3]{m^3y^3}y^2}{m^3y^3+y^3} \\ &= \lim_{y\to 0} \frac{my^3}{y^3(m^3+1)} \\ &= \lim_{y\to 0} \frac{m}{m^3+1} = \frac{m}{m^3+1}\end{aligned}$$

Since, the limit is different for each value of m the limits does not exists.

[4] **4.** State and prove the Squeeze Theorem for $f : \mathbb{R}^2 \to \mathbb{R}$.

Solution:

Squeeze Theorem: If $|f(x,y) - L| \leq B(x,y)$ for all $(x,y) \neq (a,b)$ in some neighborhood of (a,b), except possible at (a,b) and $\lim_{(x,y)\to(a,b)} B(x,y) = 0$ then $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

Proof. Since, $\lim_{(x,y)\to(a,b)} B(x,y) = 0$ we have that for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|B(x,y) - 0| < \epsilon \quad \text{whenever} \quad 0 < \|(x,y) - (a,b)\| < \delta.$$

Hence, for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x,y) - L| \leq B(x,y) < \epsilon \quad \text{whenever} \quad 0 < \|(x,y) - (a,b)\| < \delta,$$

as required.

[4] **5.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(x,y) = \left(x + \frac{1}{y^2}\right) g(xy^2), \quad y \neq 0.$$

Use the Chain Rule to show that $2xf_x - yf_y = 2f$.

Solution:

$$f_x = g(xy^2) + \left(x + \frac{1}{y^2}\right) \cdot y^2 g'(xy^2)$$
$$f_y = -\frac{2}{y^3} g(xy^2) + \left(x + \frac{1}{y^2}\right) \cdot 2xyg'(xy^2)$$

Thus,

$$\begin{aligned} 2xf_x - yf_y &= 2x\left[g(xy^2) + \left(x + \frac{1}{y^2}\right) \cdot y^2 g'(xy^2)\right] - y\left[-\frac{2}{y^3} g(xy^2) + \left(x + \frac{1}{y^2}\right) \cdot 2xyg'(xy^2)\right] \\ &= 2xg(xy^2) + 2x^2y^2g'(xy^2) + 2xg'(xy^2) + \frac{2}{y^2} g(xy^2) - 2x^2y^2g'(xy^2) - 2xg'(xy^2) \\ &= 2\left(x + \frac{1}{y^2}\right) g(xy^2) = 2f \end{aligned}$$

6. Let $f(x,y) = \ln(1 + x + 2y)$.

[3] a) Find the linear approximation $L_{(0,0)}(x,y)$ of f.

Solution: We have

$$f_x = \frac{1}{1+x+2y}, \qquad f_x(0,0) = 1$$
$$f_y = \frac{2}{1+x+2y}, \qquad f_y(0,0) = 2$$

Thus $L_{(0,0)}(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = x + 2y$.

[5] b) Use Taylor's Theorem to show that $|R_{1,(0,0)}(x,y)| \leq \frac{7}{2}(x^2+y^2)$ for $x \geq 0$, $y \geq 0$.

Solution: We have

$$f_{xx} = -(1+x+2y)^{-2}, \quad f_{xy} = -\frac{2}{(1+x+2y)^2}, \quad f_{yy} = -\frac{4}{(1+x+2y)^2}$$

Thus, we observe that since $x \geq 0$ and $y \geq 0$ that each of the second partial derivatives are decreasing, thus the maximum of each occurs at $(x,y) = (0,0)$. So, for points (x,y) near $(0,0)$

$$|f_{xx}| \leq 1, \quad |f_{xy}| \leq 2, \quad f_{yy} \leq 4.$$

Thus, by Taylor's Theorem we have

$$\begin{aligned}
|R_{1,(0,0)}| &= \frac{1}{2}\left| f_{xx}(\underline{c})(x-0)^2 + 2f_{xy}(\underline{c})(x-0)(y-0) + f_{yy}(\underline{c})(y-0)^2 \right| \\
&\leq \frac{1}{2}\left[|f_{xx}(\underline{c})|x^2 + 2|f_{xy}(\underline{c})||x||y| + |f_{yy}(\underline{c})|y^2 \right] \\
&\leq \frac{1}{2}\left[x^2 + 4|x||y| + 4y^2 \right] \\
&\leq \frac{1}{2}\left[x^2 + y^2 + 2\left(x^2 + y^2 \right) + 4x^2 + 4y^2 \right] \\
&= \frac{7}{2}[x^2+y^2],
\end{aligned}$$

as required.

7. Let $f(x,y,z) = e^{xy+z}$.

[5] a) Find the rate of change of f at the point $(1,-1,1)$ in the direction of the vector $\mathbf{u} = (4,-2,3)$.

Solution: We have $\underline{\hat{u}} = \frac{(4,-2,3)}{\|(4,-2,3)\|} = \frac{1}{\sqrt{29}}(4,-2,3)$. Thus, since f has continuous partial derivatives

$$f_x = ye^{xy+z}, \quad f_y = xe^{xy+z}, \quad f_z = e^{xy+z},$$

we get:

$$\begin{aligned} D_u f(1,-1,1) &= \nabla f(1,-1,1) \cdot \frac{1}{\sqrt{29}}(4,-2,3) \\ &= (-1,1,1) \cdot \frac{1}{\sqrt{29}}(4,-2,3) \\ &= -\frac{3}{\sqrt{29}} \end{aligned}$$

[2] b) In what direction from $(1,-1,1)$ does f change most rapidly and what is the maximum rate of change.

Solution: The direction in which f changes most rapidly is $\nabla f(1,-1,1) = (-1,1,1)$ with maximum rate of change equal to $\|(-1,1,1)\| = \sqrt{3}$.

8. Consider the function $f(x,y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

[1] a) What is the domain of f?

Solution: $\mathbb{R}^2$

[4] b) Where is f continuous on its domain?

Solution: By the continuity theorems $f(x,y)$ is continuous for all $(x,y) \neq (0,0)$....
i.e. all $(x,y) \neq (0,0)$ are typical points.

For $(x,y) = (0,0)$ we need to use the definition of continuity. So, we want to show that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$. We have

$$\begin{aligned} \left|\frac{x^3+y^3}{x^2+y^2} - 0\right| &= \frac{|x^3+y^3|}{x^2+y^2} \\ &= \frac{x^2|x| + y^2|y|}{x^2+y^2} \\ &\leq \frac{(x^2+y^2)|x| + (x^2+y^2)|y|}{x^2+y^2} \\ &= |x| + |y| = B(x,y) \end{aligned}$$

Clearly $\lim\limits_{(x,y)\to(0,0)} B(x,y) = 0$, hence

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0),$$

thus f is continuous on its domain.

[5] c) Where is f differentiable on its domain?

Solution: Observe that $f_x = \frac{x^4+3x^2y^2-2xy^3}{(x^2+y^2)^2}$ and $f_y = \frac{y^4+3x^2y^2-2yx^3}{(x^2+y^2)^2}$ are both continuous for $(x, y) \neq (0, 0)$ hence, f is differentiable for all these points.

For $(x, y) = (0, 0)$ we need to use the definition of differentiablity.

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{h^3}{h^3} = 1$$
$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{h^3}{h^3} = 1$$

Thus, both partial derivatives exist at $(0, 0)$ and $L_{(0,0)}(x, y) = x + y$. Hence,

$$R_{1,(0,0)}(x, y) = \frac{x^3 + y^3}{x^2 + y^2} - (x + y) = \frac{-xy^2 - yx^2}{x^2 + y^2}.$$

Thus, we want to show that

$$\lim_{(x,y) \to (0,0)} \frac{\left| \frac{-xy^2 - yx^2}{x^2+y^2} \right|}{\|(x, y) - (0, 0)\|} = 0.$$

But, is we approach the limit along $y = x$ we get

$$\lim_{(x,y) \to (0,0)} \frac{\left| \frac{-xy^2 - yx^2}{x^2+y^2} \right|}{\|(x, y) - (0, 0)\|} = \lim_{x \to 0} \frac{x^3}{x^3} = 1,$$

hence f is not differentiable at $(0, 0)$.

[1] d) Based on your answer in part c), what can you conclude about the continuity of f_x and f_y?

Solution: As least one of f_x and f_y are not continuous.