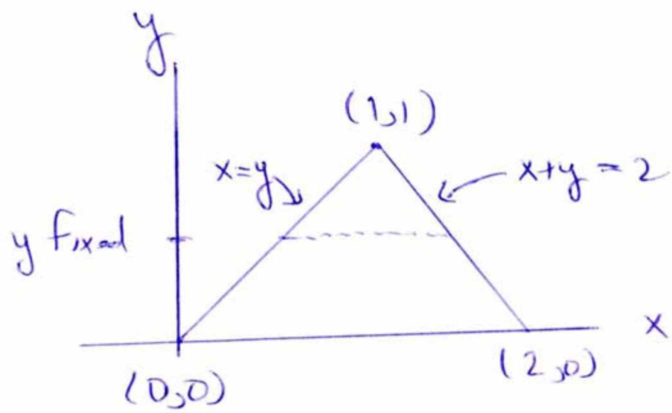
Stat 334 Midterm Solution

1. Suppose X and Y have joint pdf $f(x,y)=1$ with support defined by the triangle with vertices $(0,0)$, $(2,0)$ and $(1,1)$. Find
   a) (3 marks) the marginal pdf of Y.

The support of the joint distribution is shown is the sketch.



To find the marginal pdf of Y, we hold Y fixed and integrate over the possible values of X. From the sketch we have

$$f_2(y)=\int_{x=y}^{x=2-y} 1dx = 2-2y,\ \ 0<y<1$$

You must show the marginal support of Y that does not depend on X.

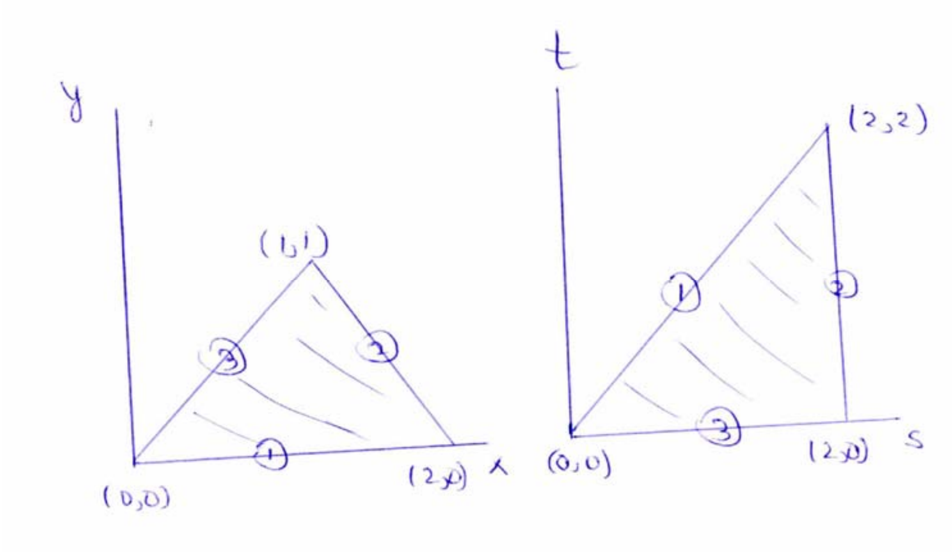
b) (3 marks) $E[XY]$

Note from the sketch it is easier to integrate first with respect to x.

$$E[XY]=\int_{y=0}^{1}\int_{x=y}^{2-y} xydxdy = \int_{y=0}^{1} y[\frac{(2-y)^2}{2}-\frac{y^2}{2}]dy = 2\int_{y=0}^{1}(y-y^2)dy = 1/3$$

c) (4 marks) the joint pdf of $S=X+Y, T=X-Y$

First we look at the support of S and T. From the sketch and mapping edges onto edges, we see that the support is the triangle $0<t<s<2$.



$1: \; y=0, 0<x<2 \Rightarrow t=s, 0<s<2$

$2: \; x+y=2, 0<y<1 \Rightarrow s=2, 0<t<2$

$3: \; y=x, 0<x<1 \Rightarrow t=0, 0<s<2$

First we look at the support of S and T. From the sketch and mapping edges onto edges, we see that the support is the triangle $0<t<s<2$.

Next we find the Jacobian. Here

$$J = \left| \det \begin{pmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{pmatrix} \right| = \left| \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right| = 2$$

Since $f(x,y)=1$ is constant, we have the joint pdf of S and T is

$$g(s,t)=\frac{1}{2}, \; 0<t<s<2$$

2. Suppose U is a discrete random variable with probability function

   $$p(u)=(\frac{1}{2})^u, \; u=1,2,...$$

   a) (3 marks) Show that the moment generating function of U is

   $$m(t)=\frac{e^t}{2-e^t}, \;\; t<\ln(2)$$

We have

$$m(t) = E[\exp\{Ut\}]$$
$$= \sum_{u=1}^{\infty} e^{ut}(1/2)^u$$
$$= \frac{e^t}{2}(1 + (\frac{e^t}{2}) + (\frac{e^t}{2})^2 + \ldots \quad \text{(an infinite geometric series)}$$
$$= \frac{e^t/2}{1 - e^t/2}, \quad e^t/2 < 1$$
$$= \frac{e^t}{2 - e^t}, \quad t < \ln(2)$$

b) (2 marks) Find $E[U]$.

$E[U] = M^{'}(0)$ and $M^{'}(t) = \frac{(2-e^t)e^t - e^t(-e^t)}{(2-e^t)^2}$ so $M^{'}(0) = \frac{1+1}{(1)^2} = 2$

3. (a) (2 marks) For any two random variables X and Y, define $E[X \mid Y]$.

First we $H(y) = E[X \mid Y = y] = \int_{\substack{all\ x\ with \\ y\ fixed}} xf(x \mid y)dx$ and then $E[X \mid Y] = h(Y)$

(b) (3 marks). Suppose $U \sim unif(0,1)$ and, given $U = u$, $Y \sim binomial(n,u)$. Find $Var[Y]$

$Var[Y] = E[Var[Y \mid U]] + Var[E[Y \mid U]]$ and we have (from simple properties of the binomial distribution)

$$E[Y \mid U = u] = nu$$
$$Var[Y \mid U = u] = nu(1-u)$$

and so

$$E[Y \mid U] = nU$$
$$Var[Y \mid U] = nU(1-U) = nU - nU^2$$

Since $U \sim unif(0,1)$ we have $f(u) = 1,\ 0 < u < 1$ and so

$$E[U] = \int_{u=0}^{1} udu = 1/2, \quad E[U^2] = \int_{u=0}^{1} u^2 du = 1/3, \ Var[U] = 1/3 - (1/2)^2 = 1/12$$

Finally

$$Var[Y] = E[Var[Y \mid U]] + Var[E[Y \mid U]]$$
$$= E[nU(1-U)] + Var[nU]$$
$$= nE[U^2] - nE[U] + n^2 Var[U]$$
$$= n/3 - n/2 + n^2/12$$
$$= n/6 + n^2/12$$