Statistics 334 Final Examination
4:00-6:30, Tuesday August 7, 2007

Some potentially useful formula

If $Z \sim N(0,1)$, the pdf of Z is $f(z) = \dfrac{\exp(-z^2/2)}{\sqrt{2\pi}}$, $-\infty < z < \infty$

If X is exponential with mean θ, the pdf of X is $f(x) = \dfrac{\exp(-x/\theta)}{\theta}$, $x > 0$

If Y is Poisson with mean μ , the pf of Y is $p(y) = \dfrac{\mu^y}{y!}e^{-u}$, $y = 0,1,2,...$

If W is Gamma with parameter $\alpha > 0$, the pdf of W is $f(w) = \dfrac{w^{\alpha-1}\exp(-w)}{\Gamma(\alpha)}$, $w > 0$

For any two random variables X and Y

$E[X] = E[E[X \mid Y]],\ Var[X] = E[Var[X \mid Y]] + Var[E[X \mid Y]]$

Questions

1. Suppose that X is a continuous random variable with probability density function (pdf) $f(x) = xe^{-x}$, $x > 0$.

a) (2 marks) Find $E[X]$.

b) (4 marks) Find the moment generating function of X . Include an interval containing 0 for which the MGF exists.

c) (4 marks) Let $T = e^{-X}$. Find the pdf of T. Don't forget to specify the support .

2. Consider the triangle $0 < y < x < 2$. We say the random variables X, Y are uniformly distributed if their joint pdf is constant on the triangle and 0 elsewhere.

a) (1 mark) Find the joint pdf of X and Y.

b) (3 marks) Find $P(X + Y > 1)$.

c) (2 marks) Find the marginal density of X.

d) (4 marks) Find the joint density of $S = X + Y$, $T = X - Y$

3. Suppose U and X are discrete random variables with joint probability function $p(u,x),\ u=1,...,x,\ x=1,2,...$.

a) (2 marks) Are X and U independent? Explain.

b) (2 marks) Carefully define $E[U \mid X]$

c) (3 marks) If $p(u,x)=\dfrac{\theta^{x-1}(1-\theta)}{x},\ u=1,...,x,\ \ x=1,2,...,\ 0<\theta<1$, find the conditional probability function of U given that $X=x$.

d) (3 marks) Evaluate $E[U]$ in terms of $E[X]$. Note that the sum of the integers from 1 to n is $n(n+1)/2$

4. Suppose $X_0, X_1, X_2, ...$ is a finite Markov chain with states $\{1,2,3\}$, $X_0=1$ and one-step transition matrix $P=\begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \end{pmatrix}$.

a) (2 marks) Using this example, explain the Markov property.

b) (2 marks) Calculate $P(X_2=1)$

c) (1 mark) Do states 1 and 3 communicate? Explain.

d) (2 marks) Are there any transient states? Explain.

e) (3 marks) Find the stationary distribution of the Markov chain.

5. Suppose that $N(t),\ t\geq 0$ with $N(0)=0$ is a Poisson process with rate parameter $\lambda>0$. Also, let T_k be the smallest value of t for which $N(t)=k$.

a) (2 marks) Find $P(N(2)=2)$

b) (2 marks) Calculate $P(N(2)=2 \mid N(1)=1)$

c) (3 marks) Find the pdf of T_1 given that $N(2)=2$.

d) (3 marks) Suppose $Y_1, Y_2, ...$ are independent normal random variables with mean μ and variance σ^2. Also $Y_1, Y_2, ...$ are independent of $N(t)$. Let $S(t)=\sum_{i=1}^{N(t)} Y_i$ if $N(t)>0$ and $S(t)=0$ if $N(t)=0$. Find $Var[S(t)]$. Be sure to explain your reasoning.

6. Suppose $Z(t)$ is a standard Brownian motion process with $Z(0)=0$.

a) (2 marks) What does it mean to say that $Z(t)$ has independent increments?

b) (2 marks) What does it mean to say that $Z(t)$ has stationary increments?

c) (3 marks) Find $Cov[Z(t), Z(t+1)]$

d) (3 marks) Calculate $P(Z(1)>0, Z(2)>0)$