Stat 334 Midterm Test   June 12, 2007

1. Suppose Y is a continuous random variable with probability density function (pdf) $f(y) = 6y(1-y),\ 0 < y < 1$. Find
   a) (2 marks) $P(1/4 \le Y \le 3/4)$
   b) (2 marks) $E[1/Y]$
   c) (3 marks) the pdf of $S = 1/Y$. Be sure to specify the support .

2. Suppose that U and V are continuous random variables with joint pdf $g(u,v) = 2e^{-(u+v)},\ 0 < v < u < \infty$. Find
   a) (2 marks) the marginal pdf of U
   b) (2 marks) the conditional pdf of V given $U = 1$
   c) (4 marks) the joint pdf of $S = U + V$, $T = U - V$

3. Suppose X is a discrete random variable with probability function $p(x) = (1-\theta)\theta^x$, $x = 0,1,2,...$ where $0 < \theta < 1$.
   a) (3 marks) Show that the moment generating function (MGF) of X is
   $$M(t) = \frac{1-\theta}{1-\theta e^t},\ t < -\ln\theta$$
   b) (2 marks) Use the MGF to find $E[X]$.
   c) (3 marks) Suppose $X_1,...,X_k$ are independent and each has the same probability function as X. Find the MGF of $T = X_1 + ... + X_k$.

4. a) (3 marks) Suppose X and Y are discrete random variables with joint probability function $p(x, y)$. Show that $E[X] = E[E[X \mid Y]]$.
   b) (3 marks) If $X \sim G(3,2)$ and, given $X = x$, $Y \sim G(x,|x|)$, find $Var[Y]$.

Stat 334 Midterm Test Solution   Spring 2007

1. Suppose Y is a continuous random variable with probability density function (pdf) $f(y) = 6y(1-y),\ 0 < y < 1$. Find

   a) (2 marks) $P(1/4 \le Y \le 3/4)$

$$P(1/4 \le Y \le 3/4) = \int_{1/4}^{3/4} 6y(1-y)dy$$
$$= 3y^2 - 2y^3 \Big|_{1/4}^{3/4}$$
$$= 11/16$$

   b) (2 marks) $E[1/Y]$

$$E[1/Y] = \int_0^1 \frac{6y(1-y)}{y} du$$
$$= \int_0^1 6(1-y)dy$$
$$= 3$$

c) (3 marks) the pdf of $S = 1/Y$. Be sure to specify the support .

Since the support of Y is $0 < y < 1$, the support of S is $s > 1$.

To find the pdf of S, first find the cdf.

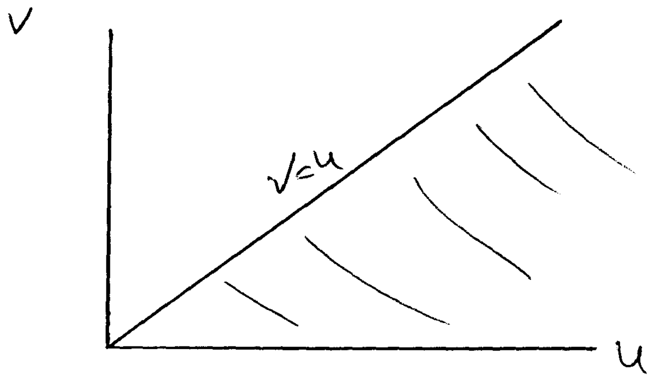
$$P(S \le s) = P(1/Y \le s)$$
$$= P(Y \ge 1/s)$$
$$= \int_{1/s}^{1} 6y(1-y)dy$$
$$= 3y^2 - 2y^3 \Big|_{1/s}^{1}$$
$$= 1 - \frac{3}{s^2} + \frac{2}{s^3}$$

The pdf is the derivative with respect to s.

$$g(s) = \frac{6}{s^3} - \frac{6}{s^4},\ s > 1$$

2. Suppose that *U* and *V* are continuous random variables with joint pdf $g(u,v) = 2e^{-(u+v)},\ 0 < v < u < \infty$. Find
   a) (2 marks) the marginal distribution of *U*.

   From the sketch, we see that the marginal pdf of *U* is



$$f_1(u) = \int_{v=0}^{u} 2e^{-(u+v)}dv$$

$$= 2e^{-u}\left(-e^{-v}\Big|_{v=0}^{u}\right)$$

$$= 2e^{-u}(1-e^{-u}),\quad u > 0$$

(be sure to include the support which only involves u)

b) (2 marks) the conditional pdf of V given $U = 1$

The conditional pdf is $f(v\,|\,u) = \dfrac{f(u,v)}{f_1(u)} = \dfrac{2e^{-(u+v)}}{2e^{-u}(1-e^{-u})} = \dfrac{e^{-v}}{(1-e^{-u})},\ 0 < v < u$. At $U = 1$,

we have $f(v\,|\,u=1) = \dfrac{e^{-v}}{(1-e^{-1})},\ 0 < v < 1$

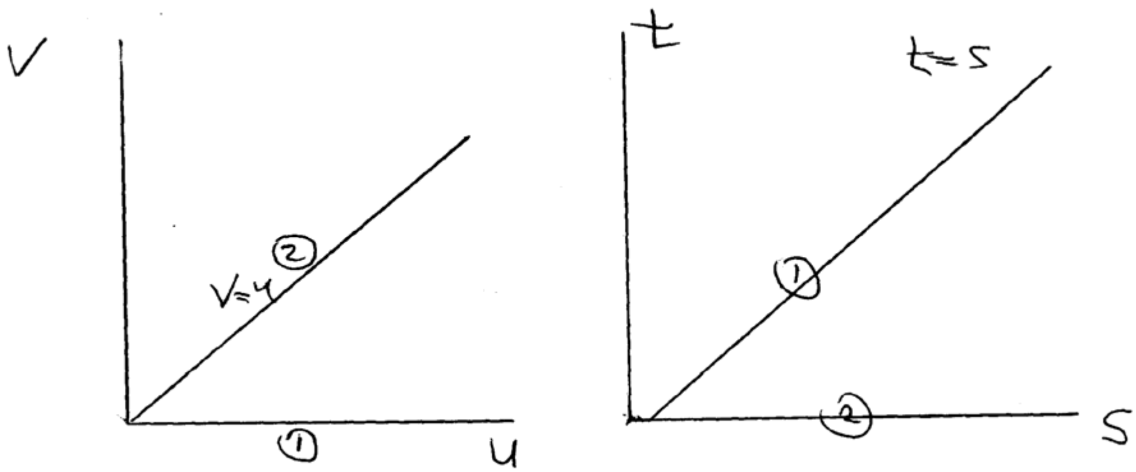
c) (4 marks) the joint pdf of $S = U + V$, $T = U - V$

We can find the support of *S* and *T* by looking at the image of the edges of the support of *U* and *V* under the transformation. See the sketch on the ext page.

$v = 0, u > 0 \Rightarrow t = s,\ s > 0$

$v = u, u > 0 \Rightarrow t = 0,\ s > 0$

The support of *S* and *T* is $0 < t < s$.



The jacobian of the transformation is $J = |\det\begin{pmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{pmatrix}| = |\det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}| = 2$ and hence the pdf of S and T is $g(s,t) = \frac{f(u(s,t), v(s,t))}{J} = \frac{2e^{-s}}{2} = e^{-s},\ 0 < t < s$

3. Suppose X is a discrete random variable with probability function $p(x) = (1-\theta)\theta^x,\ x = 0,1,2,...$ where $0 < \theta < 1$.

a) (3 marks) Show that the moment generating function (MGF) of X is

$$M(t) = \frac{1-\theta}{1-\theta e^t},\ t < -\ln\theta$$

$$\begin{aligned} M(t) &= E[e^{tX}] \\ &= \sum_{x=0}^{\infty} e^{tx}(1-\theta)\theta^x \\ &= (1-\theta)\sum_{x=0}^{\infty}(e^t\theta)^x \text{ (a geometric series that converges if } e^t\theta < 1) \\ &= \frac{1-\theta}{1-\theta e^t},\ t < -\ln\theta \end{aligned}$$

b) (2 marks) Use the MGF to find $E[X]$.

We have $M^{'}(t) = \frac{(1-\theta)\theta e^t}{(1-\theta e^t)^2}$ and $E[X] = M^{'}(0) = \frac{\theta}{1-\theta}$

c) (3 marks) Suppose $X_1, ..., X_k$ are independent and each has the same probability function as X. Find the MGF of $T = X_1 + ... + X_k$.

The MGF of *T* is

$E[e^{Tt}] = E[e^{X_1 t + X_2 t + ... + X_k t}]$

$= E[e^{X_1 t}] \times E[e^{X_2 t}] \times ... \times E[e^{X_k t}]$ (using the fact that $X_1, ..., X_k$ are independent)

$= [\frac{1-\theta}{1-\theta e^t}]^k$

4. a) (3 marks) Suppose X and Y are discrete random variables with joint probability function $p(x, y)$. Show that $E[X] = E[E[X \mid Y]]$.

First we have $h(y) = E[X \mid Y = y] = \sum_{all\, x} x p(x \mid y)$ where $p(x \mid y) = \frac{p(x, y)}{p_2(y)}$. Next we have, by definition, $E[X \mid Y] = h(Y)$ and so

$$
\begin{aligned}
E[E[X \mid Y]] &= E[h(Y)] \\
&= \sum_{all\, y} h(y) p_2(y) \\
&= \sum_{all\, y} \sum_{all\, x} x p(x \mid y) p_2(y) \\
&= \sum_{all\, y} \sum_{all\, x} x p(x, y) \\
&= E[X]
\end{aligned}
$$

b) (3 marks) If $X \sim G(3, 2)$ and, given $X = x$, $Y \sim G(x, \mid x \mid)$, find $Var[Y]$.

$Var[Y] = E[Var(Y \mid X)] + Var[E[Y \mid X]]$ and $E[Y \mid X] = X, Var[Y \mid X] = X^2$.

Hence $Var[Y] = E[X^2] + Var[X] = 4 + 3^2 + 4 = 17$